АЛГЕБРА, ГЕОМЕТРІЯ ТА ТЕОРІЯ ЙМОВІРНОСТЕЙ

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ON GROUPS IN WHICH IRREDUCIBLE SYSTEMS OF ELEMENTS FORM A MATROID

The concept of a matroid is highly versatile, offering a robust framework that captures the essence of independence in a variety of mathematical contexts. At its core, a matroid is defined by a set along with a collection of subsets (called "independent sets") that satisfy certain axioms mimicking the properties of linear independence in vector spaces. More precisely, matroid is defined as a pair (X,I), where X is a non-empty finite set, and I is a non-empty set of subsets of X that satisfies the hereditary axiom and the augmentation axiom. Such abstraction allows matroids to be applied not only in areas directly related to linear algebra but also in fields where the intuitive concept of independence or connectivity needs a formal mathematical representation. The applications of matroids in geometry, topology, combinatorial optimization, network theory, and coding theory highlight the deep interconnections between discrete mathematics and more classical fields of study, showcasing the utility and versatility of matroids as a fundamental mathematical tool.

The following methods were used to investigate the groups in which irreducible systems of elements form a matroid: the axiomatic approach, and algebraic techniques from commutative algebra and algebraic geometry, as well as combinatorial optimization and the algorithmic approach.

The paper investigates for which groups (primarily finite) G, the pair (Ĝ,I) will be a matroid. The obtained criteria of matroidality for finite and infinite abelian groups, finite nilpotent, finite symmetric, finite dihedral groups, as well as certain classes of finite matrix groups, are presented. Additionally, the non-matroidality of a wide range of finite groups has been proven, including Hamiltonian groups, groups of diagonal matrices, general and special linear groups, groups of upper triangular matrices with determinant 1, among others.

Matroids play a crucial role in algebraic research by generalizing the notion of linear independence beyond vector spaces to more abstract systems, allowing algebraic structures to be studied in a broader and more versatile context.

K e y w o r d s : ***group, matroid.***

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Introduction

The concept of a matroid generalizes the notion of the family of all linearly independent subsets of a vector space. This notion was first introduced by G. Whitney in 1935. Later, matroids found numerous applications in geometry, topology, combinatorial optimization, network theory, and coding theory (Neel, 2009).

Let us recall (Aigner, 1996; Wilson, 2010) that a *matroid* is defined as a pair $\mathbf{M} = (X, \mathcal{I})$, where X is a non-empty finite set, and $\mathcal{I}$ is a non-empty set of subsets of the set X, satisfying the following two conditions:

(M1) Hereditary Axiom: If $Y \in \mathcal{I}$ and $Z \subseteq Y$, then $Z \in \mathcal{I}$;

(M2) Augmentation Axiom: If $Y, Z \in \mathcal{I}$ and $|Z| > |Y|$, then there exists an element $z \in Z \setminus Y$ such that $Y \cup \{z\} \in \mathcal{I}$.

The set X is called the *ground set* of the matroid $\mathbf{M} = (X, \mathcal{I})$, and the members of the family $\mathcal{I}$ are called the *independent sets* of the matroid.

From axiom (M2), it follows that all maximal independent sets of a matroid have the same cardinality. Maximal independent sets of a matroid are called *bases*, and their cardinality is called the *rank* of the matroid.

Let G be a group, and let $\hat{G}$ denote the set $G \setminus \{e\}$. A subset $M \subseteq \hat{G}$, which is an independent system of generators for the subgroup $< M > \subseteq G$, is called independent. We denote the family of all independent subsets $M \subseteq \hat{G}$, augmented by the empty set, by the symbol $\mathcal{I}$.

1. Main results

The paper investigates *for which groups (primarily finite) G, the pair $(\hat{G}, \mathcal{I})$ will be a matroid.*

A group G for which the pair $(G, \mathcal{I})$ is a matroid is called *matroidal*.

It is worth noting that for any group G, the pair $(\hat{G}, \mathcal{I})$ always satisfies the hereditary axiom (M1). Therefore, it is sufficient to verify the fulfillment of the augmentation axiom (M2) only. It is also obvious that every subgroup of a matroidal group is matroidal.

2. Abelian Groups

Let p denote a prime number.

Proposition 1. *A cyclic group will be matroidal if and only if its order is a power of a prime number* (Bezushchak, 2023).

Theorem 1. *A finite abelian group G will be matroidal if and only if it is either elementary abelian or primary cyclic.*

Proof. A finite elementary abelian p-group G can be considered as a finite-dimensional vector space over the field $\mathbb{Z}_p$. Under this interpretation, a set $M \subseteq G$ will be independent if and only if it is linearly independent. Therefore, in this case, $\mathcal{I}$ coincides with the set of all linearly independent subsets of G, forming what is called a matrix matroid (Aigner, 1996; Wilson, 2010).

According to Proposition 1, a finite cyclic group will be matroidal if and only if it is a p-group.

If G is not a p-group, then it contains a cyclic subgroup that is not a p-group, and therefore not matroidal. It remains to consider the case when G is a p-group. If G is not elementary abelian, then G contains a subgroup isomorphic to the group $\mathbb{Z}_p \oplus \mathbb{Z}_{p^2}$. In this group, the sets $\{(0,p)\}$ and $\{(0,1),(1,1)\}$ belong to $\mathcal{I}$ but do not satisfy the augmentation axiom. Therefore, G contains a subgroup that is not matroidal, and thus G itself is not matroidal.

□

Corollary 1. *The class of matroidal groups is not closed under direct products.*
Proof. The groups $\mathbb{Z}_p$ and $\mathbb{Z}_{p^2}$ are matroidal, but $\mathbb{Z}_p \oplus \mathbb{Z}_{p^2}$ is not.

□

Theorem 2. *An infinite abelian group G will be matroidal if and only if it is elementary abelian or quasi-cyclic C_{p^∞}.*
Proof. If the group G is not periodic, then it contains a subgroup isomorphic to $\mathbb{Z}$. But $\mathbb{Z}$ is not matroidal, as the sets $\{1\}$ and $\{2,3\}$ belong to $\mathcal{I}$ but do not satisfy the augmentation axiom. Therefore, an infinite matroidal abelian group G must be periodic. If it contains elements of orders p and q, where p and q are distinct prime numbers, then it also contains a cyclic subgroup isomorphic to C_{pq}. According to Theorem 1, the group C_{pq} is not matroidal, so in this case, G is not matroidal.

Thus, G must be a p-group. From Theorem 1, it follows that G should not contain subgroups isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_{p^2}$ for matroidality. Therefore, for the group G to be matroidal, it needs to be either elementary abelian p-group or the group C_{p^∞}. The matroidality of an infinite elementary abelian p-group is proven in the same way as for finite ones. And the matroidality of the group C_{p^∞} follows from the fact that it has no proper independent systems, and all its proper subgroups are finite cyclic groups in which all independent systems of generators are single-element ones.

□

3. Non-abelian groups

From Proposition 1 and the fact that the group $\mathbb{Z}$ is not matroidal, it follows that a matroidal group must be periodic, and the order of each element must be a power of a prime number.

Theorem 3. *A finite nilpotent group will be matroidal if and only if it is a p-group.*
Proof. A finite nilpotent group decomposes into a direct product of its sylow subgroups. If it is not a p-group, then it contains a cyclic subgroup of order pq, which is not matroidal.

□

Proposition 2. *Hamiltonian group G is matroidal.*
Proof. For the group Q_8, the family $\mathcal{I}$ of independent subsets contains the sets $\{-1\}$ and $\{i,j\}$, which do not satisfy the augmentation axiom. Therefore, Q_8 is not matroidal. Since every Hamiltonian group contains a subgroup isomorphic to Q_8 as a direct factor, any Hamiltonian group is not matroidal.

□

Proposition 3. *The symmetric group S_n will be matroidal if and only if $n \le 3$.*
Proof. Matroidality of S_n is obvious for $n = 1$ and $n = 2$. For the group S_3, the family $\mathcal{I}$ of independent subsets consists of all one-element and two-element subsets of the set $S_3 \backslash \{e\}$, except the subset $\{(123), (132)\}$ consisting of two cycles of length 3. It can be directly verified that the augmentation axiom (M2) is satisfied. Therefore, the group S_3 is matroidal.
The group S_4 is not matroidal because the sets $\{(12), (1234)\}$ and $\{(12), (123), (124)\}$ belong to the family $\mathcal{I}$ but do not satisfy the augmentation axiom.
Finally, for $n > 4$, the group S_n is not matroidal because it contains both a subgroup isomorphic to S_4 and a subgroup isomorphic to the cyclic group C_6.

□

Theorem 4. *The dihedral group D_n will be matroidal if and only if n is a power of a prime number.*
Proof. *Necessity* of the condition is obvious: if n is not a power of a prime number, then D_n contains a cyclic subgroup C_n of rotations, which is not matroidal.
Sufficiency. Let $n = p^k$. Denote $\alpha = \frac{\pi}{p^k}$. All one-element systems in D_{p^k} are independent. Moreover, independent will be two-element systems consisting of two axial symmetries or one axial symmetry and one rotation. We will show that there are no other independent systems.
Indeed, an independent system of elements from D_{p^k} can contain at most one rotation, because in C_{p^k} all nontrivial systems are one-element ones. Moreover, an independent system of elements can contain at most two symmetries. Indeed, suppose it contains three symmetries. We can number them s_1, s_2, s_3 so that the angles between s_1 and s_2, s_2 and s_3, s_3 and s_1 are

$$n_1 p^{k_1}\alpha, \quad n_2 p^{k_2}\alpha, \quad n_3 p^{k_3}\alpha,$$

respectively, where

$$n_2 p^{k_2} + n_3 p^{k_3} + n_1 p^{k_1} = p^k,$$

n_1, n_2, n_3 are coprime with p and $k_1 = \min(k_1, k_2, k_3)$. Let O_φ denote the rotation around the center of the dihedron by the angle φ. Then the compositions $s_1 \circ s_2$ and $s_2 \circ s_3$ will be rotations

$$O_{2n_1 p^{k_1}\alpha} \quad \text{and} \quad O_{2n_2 p^{k_2}\alpha},$$

respectively.
Since n_1 is coprime to p, it is invertible in the ring $\mathbb{Z}_{p^k}$. Moreover, $k_1 \le k_2$. Therefore, there exists a natural number t such that

$$n_1 t \equiv n_2 p^{k_2 - k_1} \bmod p^k.$$

Hence,

$$(s_1 \circ s_2)^t = (O_{2n_1 p^{k_1}\alpha})^t = O_{2n_1 t p^{k_1}\alpha} = O_{2(n_2 p^{k_1 - k_2} + r p^k) p^{k_1}\alpha} =$$
$$= O_{2n_2 p^{k_2}\alpha + 2rp^{k+k_1-k}\pi} = O_{2n_2 p^{k_2}\alpha} = s_2 \circ s_3.$$

Thus, $s_3 = s_2 \circ (s_1 \circ s_2)^t$, and therefore $< s_1, s_2, s_3 > = < s_1, s_2 >$.
Finally, consider a system consisting of a rotation O_φ, where $\varphi = 2np^m\alpha$ (n and p are coprime and $m < k$), and axial symmetries s_1 and s_2. Let

$$s_1 \circ s_2 = O_{2n_1 p^{k_1}\alpha}$$

(n_1 and p are coprime and $k_1 < k$). Then if $k_1 \le m$, we have $< O_\varphi, s_1, s_2 > = < s_1, s_2 >$. If $k_1 > m$, then there exists a natural number t such that

$$nt \equiv n_1 p^{k_1 - m} \bmod p^{k-m}.$$

From the equation $s_1 \circ s_2 = O_\varphi^t$, it follows that $s_2 = s_1 \circ O_\varphi^t$, and $< O_\varphi, s_1, s_2 > = < O_\varphi, s_1 >$.

Therefore, if a system of elements contains two symmetries and a rotation, to obtain an independent system from it, either the rotation or one of the symmetries must be removed.

Thus, independent sets of elements are all single-element systems and those two-element systems consisting of two axial symmetries or one symmetry and a rotation. It is easy to see that such a collection of sets forms a matroid. This completes the proof of the theorem.

□

Proposition 4. *Let $D_n(\mathbb{Z}_p)$ be the subgroup of diagonal matrices in $GL_n(\mathbb{Z}_p)$.*

a) If $p \le 3$, then the group $D_n(\mathbb{Z}_p)$ is matroidal.

b) For any $n > 1$ and $p > 3$, the group $D_n(\mathbb{Z}_p)$ is not matroidal.

Proof. a) The group $D_n(\mathbb{Z}_2)$ is trivial, and the group $D_n(\mathbb{Z}_3)$ is an elementary abelian 2-group.

b) If $p > 3$, then $p-1$ is not a prime number. For $n > 1$, the group $D_n(\mathbb{Z}_3)$ contains the square $\mathbb{Z}_p^* \times \mathbb{Z}_p^*$ of the cyclic group $\mathbb{Z}_p^*$, which, by Theorem 1, is not matroidal.

□

Lema. *The group $UT_n(\mathbb{Z}_p)$ is generated by transvections $E + e_{i,i+1}$, where $i = 1,2,\dots,n-1$.*

Proof. We proceed by induction on n. For $n = 2$, the statement is trivial. Now, suppose the statement holds for $UT_{n-1}(\mathbb{Z}_p)$. Then in $UT_n(\mathbb{Z}_p)$, the sets of transvections

$$E + e_{i,i+1} \qquad (i = 1,2,\dots,n-2)$$

and

$$E + e_{i,i+1} \qquad (i = 2,3,\dots,n-1)$$

generate subgroups

$$\begin{pmatrix} UT_{n-1}(\mathbb{Z}_p) & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & UT_{n-1}(\mathbb{Z}_p) \end{pmatrix},$$

respectively. Therefore, the generated subgroup of transvections $E + e_{i,i+1}$ $(i = 1,2,\dots,n-1)$ contains all transvections $E + e_{i,j}$ for $i < j$, except possibly $E + e_{1,n}$. However, from the equation

$$(E + e_{1,2})(E + e_{n-1,n})(E + e_{1,2})^{-1}(E + e_{n-1,n})^{-1} = E + e_{1,n},$$

we see that it also contains this transvection. It remains to refer to the known fact [3] that

$$< (E + e_{i,i+1})_{1 \le i < n} = UT_n(\mathbb{Z}_p).$$

Lema. *The subgroup*

$$\left\{ \begin{pmatrix} 1 & 0 & x & y \\ 0 & 1 & z & u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| x, y, z, u \in \mathbb{Z}_p \right\}$$

of the group $UT_4(\mathbb{Z}_p)$ is an elementary abelian p-group.

Proof. This follows from the equation

$$\begin{pmatrix} 1 & 0 & x_1 & y_1 \\ 0 & 1 & z_1 & u_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & x_2 & y_2 \\ 0 & 1 & z_2 & u_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & x_1 + x_2 & y_1 + y_2 \\ 0 & 1 & z_1 + z_2 & u_1 + u_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

□

Theorem 5. *The group $UT_4(\mathbb{Z}_p)$ is not matroidal.*

Proof. By Lemma 1, the group $UT_4(\mathbb{Z}_p)$ has a generating set consisting of transvections

$$E + e_{12}, \qquad E + e_{23}, \qquad E + e_{34},$$

which is independent. On the other hand, the subgroup G from Lemma 2 is an elementary abelian group of order p^4, so any of its independent generating sets contains four elements. Therefore, the independent generating set of transvections of the group $UT_4(\mathbb{Z}_p)$ and the independent generating set of subgroups G do not satisfy the augmentation axiom.

□

Corollary 2. *For $n \ge 4$, each of the groups $GL_n(\mathbb{Z}_p)$, $SL_n(\mathbb{Z}_p)$, $T_n(\mathbb{Z}_p)$, $UT_n(\mathbb{Z}_p)$ is not matroidal.*

Proof. Each of these groups contains a subgroup isomorphic to $UT_4(\mathbb{Z}_p)$.

□

Proposition 5. *For $n \le 3$, the group $UT_n(\mathbb{Z}_p)$ is matroidal.*

Proof. The group $UT_1(\mathbb{Z}_p)$ is trivial, and

$$UT_2(\mathbb{Z}_p) \cong \mathbb{Z}_p.$$

Therefore, they are matroidal.

Consider the case $n = 3$. The group $UT_3(\mathbb{Z}_2)$ is isomorphic to D_4, hence it is matroidal. Now, let $p > 2$. In the group $UT_3(\mathbb{Z}_p)$, the independent systems consist only of single-element subsets and some two-element ones $\{A, B\}$, where $< A > \neq < B >$. Indeed, if $A, B \in UT_3(\mathbb{Z}_p)$ and $< A > \neq < B >$, then either $AB = BA$ (and then $< A, B > \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$), or $AB \neq BA$ (and then $< A, B > \cong UT_3(\mathbb{Z}_p)$). Therefore, if an independent system contains noncommuting elements, it contains only two elements. Moreover, it cannot contain three pairwise commuting elements, because it has order p^3 and is noncommutative.

Thus, the family $\mathcal{I}$ consists of all single-element subsets and some two-element ones $\{A, B\}$ such that $< A > \neq < B >$. Since the augmentation axiom is satisfied, the group $UT_3(\mathbb{Z}_p)$ is matroidal.

□

The matroidality of the remaining matrix groups over $\mathbb{Z}_p$ for $n \leq 3$ needs to be considered. Over the field $\mathbb{Z}_2$,

$$T_n(\mathbb{Z}_2) = UT_n(\mathbb{Z}_2) \quad \text{and} \quad SL_n(\mathbb{Z}_2) = GL_n(\mathbb{Z}_2).$$

Therefore, for $\mathbb{Z}_2$, only the case of the group $GL_n(\mathbb{Z}_2)$ needs to be considered.

Proposition 6. *The group $GL_2(\mathbb{Z}_2)$ is matroidal, while $GL_3(\mathbb{Z}_2)$ is not.*

Proof. The matroidality of the group $GL_2(\mathbb{Z}_2)$ follows from the isomorphism $GL_2(\mathbb{Z}_2) \cong S_3$ and Proposition 3.

$GL_3(\mathbb{Z}_2)$ is a simple group of order 168. From the Cayley graph of the group permutations on the cosets, it follows that a simple group of order 168 cannot have subgroups of index ≤ 5.

On the other hand, $GL_3(\mathbb{Z}_2)$ contains more than one subgroup of order 7. Therefore, any two elements of order 7 in $GL_3(\mathbb{Z}_2)$, which are not powers of each other, generate the entire group $GL_3(\mathbb{Z}_2)$.

Let's consider the matrices

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

It is obvious that $C \notin < A, B >$. Furthermore, since the set of matrices of the form

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix} \quad \left(\text{or } \begin{pmatrix} * & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix}\right)$$

in $GL_3(\mathbb{Z}_2)$ is closed under multiplication, $B \notin < A, C >$ (or $A \notin < B, C >$). Therefore, the set of elements $\{A, B, C\}$ is independent. However, this set and any set of generators of the group $GL_3(\mathbb{Z}_2)$ with two elements of order 7 do not satisfy the augmentation axiom.

□

Proposition 7. *All groups $GL_n(\mathbb{Z}_3)$, $SL_n(\mathbb{Z}_3)$, and $T_n(\mathbb{Z}_3)$ for $n = 2,3$ are not matroidal.*

Proof. The group $T_2(\mathbb{Z}_3)$ contains a cyclic subgroup generated by the matrix $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ of order 6, hence it is not matroidal. Each of the groups $T_2(\mathbb{Z}_3)$, $GL_n(\mathbb{Z}_3)$ $(n = 2,3)$, and $SL_3(\mathbb{Z}_3)$ contains a subgroup isomorphic to $T_2(\mathbb{Z}_3)$, hence they are not matroidal either. Finally, in the group $SL_2(\mathbb{Z}_3)$, the subgroup generated by the matrices $\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ is isomorphic to the quaternion group Q_8, so $SL_2(\mathbb{Z}_3)$ is also not matroidal.

□

Proposition 8. *Let $p > 3$. None of the groups $GL_n(\mathbb{Z}_p)$, $SL_n(\mathbb{Z}_p)$, and $T_n(\mathbb{Z}_p)$ for $n = 2,3$ are matroidal.*

Proof. The group $T_2(\mathbb{Z}_p)$ contains a subgroup $D_2(\mathbb{Z}_p)$, which according to Proposition 4.b is not matroidal. Therefore, $T_2(\mathbb{Z}_p)$ is also not matroidal. In turn, $T_3(\mathbb{Z}_p)$ contains a subgroup isomorphic to $T_2(\mathbb{Z}_p)$, hence $T_3(\mathbb{Z}_p)$ is not matroidal as well.

The group $SL_2(\mathbb{Z}_p)$ contains an abelian subgroup

$$\left\{\begin{pmatrix} \varepsilon & a \\ 0 & \varepsilon \end{pmatrix} \middle| \ \varepsilon = \pm 1, \ a \in \mathbb{Z}_p\right\},$$

which has order $2p$, and therefore, by Theorem 2, it is not matroidal. Thus, $SL_2(\mathbb{Z}_p)$ is not matroidal. In turn, each of the groups $SL_3(\mathbb{Z}_p)$, $GL_2(\mathbb{Z}_p)$, and $GL_3(\mathbb{Z}_p)$ contains a subgroup isomorphic to $SL_2(\mathbb{Z}_p)$, so they are not matroidal either.

□

Discussion and conclusions

The study of matroids remains a vibrant area of research with many open problems. Establishing criteria for matroidality for various algebraic objects is a challenging yet promising task. Several results characterize matroidality in specific contexts, but a universal set of criteria remains elusive. In this article, we examined the criteria of matroidality for particular classes of finite and infinite groups.

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ПРО ГРУПИ, У ЯКИХ НЕЗВІДНІ СИСТЕМИ ЕЛЕМЕНТІВ УТВОРЮЮТЬ МАТРОЇД

Концепція матроїда є дуже універсальною, пропонуючи надійну основу, що відображає суть незалежності в різноманітних математичних контекстах. По суті, матроїд визначається множиною разом із сукупністю підмножин (т. зв. "незалежними множинами"), які задовольняють певні аксіоми, що імітують властивості лінійної незалежності у векторних просторах. Точніше, матроїд визначають як пару (X,I), де X – це непорожня скінченна множина, а I – непорожня множина підмножин X, що задовольняє аксіому спадковості й аксіому доповнення. Така абстракція дає змогу застосовувати матроїди не тільки в галузях, безпосередньо пов'язаних з лінійною алгеброю, але

й у тих, де інтуїтивне поняття незалежності або зв'язності потребує формального математичного зображення. Застосування матроїдів у геометрії, топології, комбінаторній оптимізації, теорії мереж і теорії кодування підкреслює глибокий взаємозв'язок між дискретною математикою та більш класичними галузями дослідження, демонструючи корисність й універсальність матроїдів як фундаментального математичного інструменту.

У пропонованій роботі з'ясовано, для яких груп (насамперед скінченних) G-пара $(\hat{G},I)$ ***буде матроїдом. Отримано критерії матроїдальності для скінченних і нескінченних абелевих груп, для скінченних нільпотентних, скінченних симетричних і скінченних діедральних груп та окремих класів скінченних матричних груп. А також доведено нематроїдальність цілої низки скінченних груп, серед яких гамільтонові групи, групи діагональних матриць, загальні та спеціальні лінійні групи, групи верхніх трикутних матриць з визначником 1 тощо.***

Для дослідження груп, у яких незвідні системи елементів формують матроїд, були застосовані такі підходи: аксіоматичний підхід та алгебраїчні техніки з комутативної алгебри й алгебраїчної геометрії, а також комбінаторна оптимізація й алгоритмічний підхід.

Матроїди відіграють важливу роль в алгебраїчних дослідженнях, узагальнюючи поняття лінійної незалежності за межами векторних просторів на більш абстрактні системи, що дає змогу вивчати алгебричні структури в ширшому й універсальнішому контексті.

К л ю ч о в і с л о в а : *група, матроїд.*

Автори заявляють про відсутність конфлікту інтересів. Спонсори не брали участі в розробленні дослідження; у зборі, аналізі чи інтерпретації даних; у написанні рукопису; в рішенні про публікацію результатів.

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