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Асимптотична нормальність оцінок параметрів змішаного дробового броунівського руху

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Asymptotically normal estimation of parameters of mixed fractional Brownian motion

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У роботі досліджується модель змішаного дробового броунівського руху вигляду $X_t = \sigma W_t + \kappa B_t^H$, що складається зі стандартного броунівського руху W та дробового броунівського руху B^H з індексом Херста H. Розглядаються строго консистентні оцінки невідомих параметрів моделі (H, κ, σ) на основі рівновіддалених спостережень траєкторії. Доведено спільну асимптотичну нормальність оцінок для $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{4})$.

Ключові слова: дробовий броунівський рух, вінерівський процес, змішана модель, параметричне оцінювання, асимптотичний розподіл.

We investigate the mixed fractional Brownian motion of the form $X_t = \sigma W_t + \kappa B_t^H$, driven by a standard Brownian motion W and a fractional Brownian motion B^H with Hurst parameter H. We consider strongly consistent estimators of unknown model parameters (H, κ, σ) based on the equidistant observations of a trajectory. Joint asymptotic normality of these estimators is proved for $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{4})$.

Key Words: Fractional Brownian motion, Wiener process, mixed model, parameter estimation, asymptotic distribution.

1 Introduction

Many real-world processes that evolve over time are traditionally mathematically modeled using the standard Brownian motion. Still, multiple studies show that some time series possess properties of self-similarity, long-range dependence and complex correlation structures [1] and thus cannot be modeled properly by the usage of the Brownian motion only. Instead, a fractional Brownian motion can be used as it has correlated increments which implies short-range ($H < 1/2$) or long-range ($H > 1/2$) dependence [2]. The mixed fractional Brownian motion combines properties of both standard and fractional Brownian motions; this makes it a suitable model for many financial applications. It was first proposed by Cheridito [3]. There are already existing practical applications of this model, see, e.g., [4]. Its properties can be found in [5]. In the present paper we consider the following mixed fractional Brownian motion:

$$X_t = \sigma W_t + \kappa B_t^H,\ t \geqslant 0, \tag{1}$$

where W is a Wiener process, B^H is a fractional Brownian motion with Hurst index H, B^H is independent of W. We aim to estimate unknown parameters $(H, \kappa, \sigma)^\top$ based on observed $\{X_{kh}, k = 0, 1, 2, \dots\}$, $h > 0$.

In the present paper, following [6], we introduce the next statistics

$$\begin{aligned} \xi_N &:= \frac{1}{N}\sum_{k=0}^{N-1}\left(X_{(k+1)h} - X_{kh}\right)^2, \\ \eta_N &:= \frac{1}{N}\sum_{k=0}^{N-1}\left(X_{(k+1)h} - X_{kh}\right)\times \\ &\qquad \times\left(X_{(k+2)h} - X_{(k+1)h}\right), \\ \zeta_N &:= \frac{1}{N}\sum_{k=0}^{N-1}\left(X_{(k+2)h} - X_{kh}\right)\times \\ &\qquad \times\left(X_{(k+4)h} - X_{(k+2)h}\right). \end{aligned} \tag{2}$$

Let us denote

$$\tau_N = (\xi_N, \eta_N, \zeta_N)^\top. \tag{3}$$

According to [6, Lemma 3.2], for any $H \in (0,1)$,

$$\tau_N \to (\mathsf{E}\,\xi_1, \mathsf{E}\,\eta_1, \mathsf{E}\,\zeta_1)^\top = \\ = \left(\sigma^2 h + \kappa^2 h^{2H}, \kappa^2 h^{2H}\left(2^{2H-1}-1\right), \\ \kappa^2 h^{2H} 2^{2H}\left(2^{2H-1}-1\right)\right)^\top =: \tau_0$$

a. s., as $N \to \infty$. Then, similarly to [6], for $H \in (0,\frac12) \cup (\frac12, 1)$, we can consider the following strongly consistent estimators for the parameters $(H, \kappa^2, \sigma^2)^\top$:

$$\begin{aligned} \widehat{H}_N &= \frac12 \log_{2+} \frac{\zeta_N}{\eta_N}, \\ \hat\kappa^2_N &= \frac{\eta_N}{h^{2\widehat{H}_N}\left(2^{2\widehat{H}_N - 1} - 1\right)}, \\ \hat\sigma^2_N &= \frac1h \left(\xi_N - \hat\kappa^2_N h^{2\widehat{H}_N}\right) \end{aligned} \tag{4}$$

with

$$\log_{2+} x = \begin{cases} \log_2 x, & \text{if } x > 0, \\ 0, & \text{if } x \leqslant 0. \end{cases}$$

(Evidently, the model is non-identifiable in the case $H = \frac12$, so we should exclude this value.) The goal of the present paper is to study joint asymptotic normality of these estimators. It worth mentioning that asymptotic normality does not hold for $H > \frac34$, which is typical for models with fractional Brownian motion, see, e.g., [7]. The estimators remain strongly consistent, but have non-Gaussian asymptotic distribution.

The paper is organized as follows. In Section 2 we prove the joint asymptotic normality of statistics (2) and evaluate their asymptotic covariance matrix. Thereafter, in Section 3 we obtain the main result on asymptotic normality of (4) by applying the delta method. Section 4 we investigate the behavior of the estimators numerically, using Monte Carlo simulations.

We use the following notation. The symbol E denotes expectation and acts as an operator on the total product of quantities, **cov** stands for the covariance of two random variables and for the covariance matrix of a random vector. The upper index $\top$ denotes transposition. In the paper, all the vectors are column ones. Weak convergence in distribution is denoted by $\xrightarrow{\mathrm{d}}$.

2 Asymptotic normality of vector $(\xi_N, \eta_N, \zeta_N)^\top$

We will start by studying the asymptotic properties of the vector $(\xi_N, \eta_N, \zeta_N)^\top$. Let us denote the increment

$$\Delta X_k := X_{(k+1)h} - X_{kh} = \sigma \Delta W_k + \kappa \Delta B_k^H.$$

It is well-known that $\{\Delta X_k\}$ is a stationary Gaussian sequence with the following autocovariance function

$$\tilde\rho(i) = \mathbf{cov}(\Delta X_0, \Delta X_i) = \sigma^2\,\mathbf{cov}\left(\Delta W_0, \Delta W_i\right) + \\ + \kappa^2\,\mathbf{cov}\left(\Delta B_0^H, \Delta B_i^H\right) = \\ = \sigma^2 h \mathbb{1}_{\{i=0\}} + \kappa^2 h^{2H} \rho(i), \quad i \in \mathbb{Z}, \tag{5}$$

where

$$\rho(i) := \frac12 \left(|i+1|^{2H} - 2|i|^{2H} + |i-1|^{2H}\right) \tag{6}$$

denotes the autocovariance function of the stationary sequence $\{B_{k+1}^H - B_k^H, k \geqslant 0\}$, which is known as fractional Gaussian noise.

The following statement provides some important properties of sequences $\tilde\rho(i)$ and $\rho(i)$, $i \in \mathbb{Z}$, defined by (5) and (6) respectively.

Lemma 1. *Let $H \in (0, \frac34)$. Then the following statements hold*

1) $\rho(i) \sim H(2H-1)|i|^{2H-2}$ as $|i| \to \infty$,

2) $\sum_{i=-\infty}^{\infty} \rho^2(i) < \infty$,

3) For all $\alpha, \beta \in \mathbb{Z}$,

$$\sum_{i=-\infty}^{+\infty} \tilde\rho(i+\alpha)\tilde\rho(i+\beta) < \infty. \tag{7}$$

Proof. First two statements are well known, see e. g., [7]. In order to prove the third one, we observe that

$$\sum_{i=-\infty}^{\infty} \tilde\rho^2(i) = \sigma^4 h^2 + 2\sigma^2\kappa^2 h^{2H+1} + \\ + \kappa^4 h^{4H} \sum_{i=-\infty}^{\infty} \rho^2(i) < \infty,$$

by the second statement. Then, using the Cauchy–Schwarz inequality, we get for all $\alpha, \beta \in \mathbb{Z}$,

$$\sum_{i=-\infty}^{+\infty} \tilde\rho(i+\alpha)\tilde\rho(i+\beta) \leqslant \\ \leqslant \sqrt{\sum_{i=-\infty}^{+\infty} \tilde\rho^2(i+\alpha) \cdot \sum_{i=-\infty}^{+\infty} \tilde\rho^2(i+\beta)} = \\ = \sum_{i=-\infty}^{+\infty} \tilde\rho^2(i) < \infty,$$

where the equality follows by substitutions $i' = i + \alpha$ and $i' = i + \beta$. □

Now we are ready to prove the main result of this section.

Theorem 2.1. *Let $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{4})$. The vector τ_N defined by (3) is asymptotically normal, namely*

$$\sqrt{N}(\tau_N - \tau_0) = \sqrt{N} \begin{pmatrix} \xi_N - \mathsf{E}\,\xi_N \\ \eta_N - \mathsf{E}\,\eta_N \\ \zeta_N - \mathsf{E}\,\zeta_N \end{pmatrix} \xrightarrow{\mathrm{d}} \mathcal{N}(\vec{0}, \widetilde{\Sigma})$$

with the asymptotic covariance matrix $\widetilde{\Sigma}$, which can be presented explicitly as

$$\widetilde{\Sigma} = \begin{pmatrix} \widetilde{\Sigma}_{11} & \widetilde{\Sigma}_{12} & \widetilde{\Sigma}_{13} \\ \widetilde{\Sigma}_{12} & \widetilde{\Sigma}_{22} & \widetilde{\Sigma}_{23} \\ \widetilde{\Sigma}_{13} & \widetilde{\Sigma}_{23} & \widetilde{\Sigma}_{33} \end{pmatrix}$$

where

$$\widetilde{\Sigma}_{11} = 2 \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i)^2,$$

$$\widetilde{\Sigma}_{22} = \sum_{i=-\infty}^{\infty} \tilde{\rho}(i)\Big(\tilde{\rho}(i) + \tilde{\rho}(i+2)\Big),$$

$$\widetilde{\Sigma}_{33} = \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i)\Big(6\tilde{\rho}(i) + 8\tilde{\rho}(i+1) + 3\tilde{\rho}(i+2) + 4\tilde{\rho}(i+3) + 6\tilde{\rho}(i+4) + 4\tilde{\rho}(i+5) + \tilde{\rho}(i+6)\Big),$$

$$\widetilde{\Sigma}_{12} = 2 \sum_{i=-\infty}^{\infty} \tilde{\rho}(i)\tilde{\rho}(i+1),$$

$$\widetilde{\Sigma}_{13} = 2 \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i)\Big(\tilde{\rho}(i+1) + 2\tilde{\rho}(i+2) + \tilde{\rho}(i+3)\Big),$$

$$\widetilde{\Sigma}_{23} = \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i)\Big(\tilde{\rho}(i) + 2\tilde{\rho}(i+1) + 2\tilde{\rho}(i+2) + 2\tilde{\rho}(i+3) + \tilde{\rho}(i+4)\Big).$$

Proof. The proof consists of two parts: in the first part we compute the asymptotic covariance matrix $\widetilde{\Sigma}$, while the second part contains the proof of asymptotic normality.

Part 1: Identification of the asymptotic covariance matrix. First, we will find the explicit form of covariance matrix $\widetilde{\Sigma}$ by evaluating convergence limits as $N \to \infty$ for the following variances and covariances: $N\,\mathbf{Var}(\xi_N)$, $N\,\mathbf{Var}(\eta_N)$, $N\,\mathbf{Var}(\zeta_N)$, $N\,\mathbf{cov}(\xi_N, \eta_N)$, $N\,\mathbf{cov}(\xi_N, \zeta_N)$, $N\,\mathbf{cov}(\eta_N, \zeta_N)$.

1.1 Evaluation of convergence limit for $N\,\mathbf{Var}(\xi_N)$. Using Isserlis' theorem[1], and stationarity of the sequence $\{\Delta X_k\}$, we can write

$$\mathbf{cov}\left((\Delta X_k)^2, (\Delta X_j)^2\right) = 2\tilde{\rho}(k-j)^2.$$

Therefore

$$N\,\mathbf{Var}(\xi_N) = \frac{1}{N}\mathbf{Var}\left(\sum_{k=0}^{N-1} (\Delta X_k)^2\right) = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} \mathbf{cov}\left((\Delta X_k)^2, (\Delta X_j)^2\right) = \frac{2}{N} \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} \tilde{\rho}(k-j)^2. \tag{8}$$

Further, by rearranging sums, we get

$$N\,\mathbf{Var}(\xi_N) = \frac{2}{N} \sum_{k=0}^{N-1} \sum_{i=k-N+1}^{k} \tilde{\rho}(i)^2 = \frac{2}{N} \sum_{i=-N+1}^{0} \sum_{k=0}^{N-1+i} \tilde{\rho}(i)^2 + \frac{2}{N} \sum_{i=1}^{N-1} \sum_{k=i}^{N-1} \tilde{\rho}(i)^2 = 2 \sum_{i=-N+1}^{N-1} \left(1 - \frac{|i|}{N}\right) \tilde{\rho}(i)^2 \to 2 \sum_{i=-\infty}^{\infty} \tilde{\rho}(i)^2, \quad \text{as } N \to \infty. \tag{9}$$

where the passage to the limit can be justified by the dominated convergence theorem due to Lemma 1.

1.2. Evaluation of convergence limit for $N\,\mathbf{cov}(\xi_N, \eta_N)$. Write

$$N\,\mathbf{cov}(\xi_N, \eta_N) = \frac{1}{N}\mathbf{cov}\left(\sum_{k=0}^{N-1} (\Delta X_k)^2, \sum_{j=0}^{N-1} \Delta X_j \Delta X_{j+1}\right) = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} \mathbf{cov}\left((\Delta X_k)^2, \Delta X_j \Delta X_{j+1}\right).$$

By Isserlis' theorem,

$$\mathbf{cov}\left((\Delta X_k)^2, \Delta X_j \Delta X_{j+1}\right) = 2\tilde{\rho}(j-k)\tilde{\rho}(j-k+1).$$

[1]Isserlis' theorem [8]: if (X_1, X_2, X_3, X_4) is a zero-mean multivariate normal random vector, then $\mathsf{E}(X_1X_2X_3X_4) = \mathsf{E}\,X_1X_2\,\mathsf{E}\,X_3X_4 + \mathsf{E}\,X_1X_3\,\mathsf{E}\,X_2X_4 + \mathsf{E}\,X_1X_4\,\mathsf{E}\,X_2X_3$. In particular, $\mathbf{cov}(X_1^2, X_2^2) = 2\,\mathbf{cov}(X_1, X_2)^2$.

Therefore arguing as in (9), we obtain

$$N\,\mathbf{cov}(\xi_N,\eta_N) = \\ = \frac{2}{N}\sum_{j=0}^{N-1}\sum_{k=0}^{N-1}\tilde\rho(j-k)\tilde\rho(j-k+1) = \\ = \frac{2}{N}\sum_{j=0}^{N-1}\sum_{i=j-N+1}^{j}\tilde\rho(i)\tilde\rho(i+1) = \\ = 2\sum_{i=-N+1}^{N-1}\left(1-\frac{|i|}{N}\right)\tilde\rho(i)\tilde\rho(i+1) \to \\ \to 2\sum_{i=-\infty}^{\infty}\tilde\rho(i)\tilde\rho(i+1),$$

as $N\to\infty$, where the last series converges according to the Lemma 1.

1.3. Evaluation of convergence limit for $N\,\mathbf{Var}(\eta_N)$. We have

$$N\,\mathbf{Var}(\eta_N) \\ = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\mathbf{cov}(\Delta X_k\Delta X_{k+1},\Delta X_j\Delta X_{j+1}).$$

By Isserlis' Theorem,

$$\mathbf{cov}(\Delta X_k\Delta X_{k+1},\Delta X_j\Delta X_{j+1}) = \\ = \tilde\rho(k-j)^2+\tilde\rho(k-j+1)\tilde\rho(k-j-1),$$

Therefore

$$\mathbf{Var}(\eta_N) = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\tilde\rho(k-j)^2 + \\ + \frac{1}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\tilde\rho(k-j+1)\tilde\rho(k-j-1). \quad (10)$$

According to (8) and (9), the first term in the right hand side of (10)

$$\frac{1}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\tilde\rho(k-j)^2 = \\ = \frac{N}{2}\mathbf{Var}(\xi_N) \to \sum_{i=-\infty}^{\infty}\tilde\rho(i)^2, \quad N\to\infty.$$

The second term can be treated similarly to (9) as follows:

$$\frac{1}{N}\sum_{j=0}^{N-1}\sum_{k=0}^{N-1}\tilde\rho(k-j+1)\tilde\rho(k-j-1) = \\ = \frac{1}{N}\sum_{j=0}^{N-1}\sum_{i=j-(N-1)}^{j}\tilde\rho(i-1)\tilde\rho(i+1) = \\ = \sum_{i=-(N-1)}^{N-1}\left(1-\frac{|i|}{N}\right)\tilde\rho(i-1)\tilde\rho(i+1) \to \\ \to \sum_{i=-\infty}^{\infty}\tilde\rho(i-1)\tilde\rho(i+1) = \\ = \sum_{i=-\infty}^{\infty}\tilde\rho(i)\tilde\rho(i+2), \quad \text{as } N\to\infty. \quad (11)$$

Combining (10)–(11), we arrive at

$$\lim_{N\to\infty}N\,\mathbf{Var}(\eta_N) = \sum_{i=-\infty}^{\infty}\tilde\rho(i)\Big(\tilde\rho(i)+\tilde\rho(i+2)\Big).$$

Convergency limits for $N\,\mathbf{cov}(\xi_N,\zeta_N)$, $N\,\mathbf{cov}(\eta_N,\zeta_N)$, $N\,\mathbf{Var}(\zeta_N)$ are evaluated in a similar way.

Part 2: Proof of asymptotic normality. Let us define $Y_k = \left(Y_k^{(1)},Y_k^{(2)},Y_k^{(3)}\right)^\top$ by

$$Y_k^{(1)} := \Delta X_k, \quad Y_k^{(2)} := \Delta X_{k+1}, \\ Y_k^{(3)} := \Delta X_{k+2}+\Delta X_{k+3}. \quad (12)$$

Then

$$\xi_N = \frac{1}{N}\sum_{k=0}^{N-1}\left(Y_k^{(1)}\right)^2, \quad \eta_N = \frac{1}{N}\sum_{k=0}^{N-1}Y_k^{(1)}Y_k^{(2)}, \\ \zeta_N = \frac{1}{N}\sum_{k=0}^{N-1}\left(Y_k^{(1)}+Y_k^{(2)}\right)Y_k^{(3)}.$$

We shall prove the convergence of vector $(\xi_N,\eta_N,\zeta_N)^\top$ with the help of the Cramér–Wold device. Let the parameters α, β, $\gamma\in\mathbb{R}$ be any fixed ones satisfying the condition $\alpha^2+\beta^2+\gamma^2\neq 0$. We introduce the function

$$f(y) = \alpha y_1^2+\beta y_1y_2+\gamma(y_1+y_2)y_3, \\ y = (y_1,y_2,y_3)^\top\in\mathbb{R}^3,$$

so that

$$\alpha\xi_N+\beta\eta_N+\gamma\zeta_N = \frac{1}{N}\sum_{k=0}^{N-1}f(Y_k).$$

Thus, we need to prove that the sequence

$$\sqrt{N}\Big(\alpha(\xi_N-\mathsf{E}\,\xi_N)+\beta(\eta_N-\mathsf{E}\,\eta_N)+ \\ +\gamma(\zeta_N-\mathsf{E}\,\zeta_N)\Big) = \frac{1}{\sqrt N}\sum_{k=0}^{N-1}\Big(f(Y_k)-\mathsf{E}\,f(Y_k)\Big) \quad (13)$$

converges to a normal distribution. This fact can be established by application of the Breuer–Major theorem for stationary vectors [9, Theorem 4], see also [10, Theorem 1.1]. In order to apply this theorem, it suffices to verify the following condition:

$$\sum_{j\in\mathbb{Z}} \left| r^{(p,l)}(j) \right|^q < \infty, \quad \text{for all } p,l \in \{1,2,3\}. \tag{14}$$

where

$$r^{(p,l)}(k) := \mathbf{cov}\left(Y_1^{(p)}, Y_{1+k}^{(l)}\right), \quad k\in\mathbb{Z},$$

and q is the Hermite rank[2] of f with respect to Y_1.

It is not hard to see that this Hermite rank $q \geqslant 2$. Indeed, note that $f(Y_1)$ is a second-order polynomial of zero-mean normally distributed random variables $Y_1^{(1)}, Y_1^{(2)}, Y_1^{(3)}$ in which only terms of the second order are present, namely $(Y_1^{(1)})^2$, $Y_1^{(1)}Y_1^{(2)}$, and $Y_1^{(1)}Y_1^{(3)}$. Therefore, according to Isserlis' theorem, we will have that the expected value of the product of $f(Y_1)$ and any of $Y_1^{(t)}$, $t \in \{1,2,3\}$, will be equal to zero. All the obtained terms will have the form $\mathsf{E}\, G_1G_2G_3$, where $(G_1,G_2,G_3)^\top$ is a zero-mean multivariate normal random vector and according to the Isserlis' theorem $\mathsf{E}\, G_1G_2G_3 = 0$. Therefore, the expected value of the product of $f(Y_1)$ and any first-order polynomial of $\{Y_1\}$ (a linear combination of $Y_1^{(1)}, Y_1^{(2)}, Y_1^{(3)}$) will be equal to zero, and therefore the Hermite rank q of the function f with respect to Y_1 cannot be equal to 1.

Since $q \geqslant 2$, we easily see that in order to verify the condition (14), it suffices to prove that

$$\sum_{j\in\mathbb{Z}} \left| r^{(p,l)}(j) \right|^2 < \infty, \quad \text{for all } p,l \in \{1,2,3\}. \tag{15}$$

In turn, (15) follows from Lemma 1, since using (12) and the definition of $\tilde\rho$, we can represent $r^{(p,l)}$ via $\tilde\rho$ as follows:

$$\begin{gathered} r^{(1,1)}(k) = r^{(2,2)}(k) = \tilde\rho(k), \\ r^{(1,2)}(k) = \tilde\rho(k+1), \quad r^{(2,1)}(k) = \tilde\rho(k-1), \\ r^{(1,3)}(k) = \tilde\rho(k+2) + \tilde\rho(k+3), \\ r^{(3,1)}(k) = \tilde\rho(k-2) + \tilde\rho(k-3), \\ r^{(2,3)}(k) = \tilde\rho(k+1) + \tilde\rho(k+2), \\ r^{(3,2)}(k) = \tilde\rho(k-1) + \tilde\rho(k-2), \\ r^{(2,3)}(k) = \tilde\rho(k+1) + 2\tilde\rho(k) + \tilde\rho(k-1). \end{gathered}$$

Indeed, for example the series

$$\begin{aligned} &\sum_{k=-\infty}^{+\infty} \left(r^{(1,3)}(k)\right)^2 = \\ &= \sum_{k=-\infty}^{+\infty} (\tilde\rho(k+2)+\tilde\rho(k+3))^2 = \\ &= \sum_{k=-\infty}^{+\infty} \tilde\rho(k+2)^2 + 2\sum_{k=-\infty}^{+\infty} \tilde\rho(k+2)\times \\ &\times \tilde\rho(k+3) + \sum_{k=-\infty}^{+\infty} \tilde\rho(k+3)^2, \end{aligned}$$

converges as a finite sum of series convergent according to Lemma 1. Similarly, other series in (15) are also finite.

Therefore, the assumptions of the Breuer–Major theorem for stationary vectors are satisfied, whence the desired weak converge of (13) to a zero-mean normal distribution follows. □

3 Asymptotic normality of vector $(\widehat{H}_N, \hat\kappa_N^2, \hat\sigma_N^2)^\top$

In this section, we provide the main result on asymptotic properties of our estimator (4). Let us denote

$$\begin{gathered} \vartheta = (H, \kappa^2, \sigma^2)^\top, \\ \hat\vartheta_N = (\widehat{H}_N, \hat\kappa_N^2, \hat\sigma_N^2)^\top,\ N\in\mathbb{N}. \end{gathered} \tag{16}$$

Theorem 3.1. *Let $H \in (0,\frac12)\cup(\frac12,\frac34)$. The estimator $\hat\vartheta_N$ is asymptotically normal, namely*

$$\sqrt{N}\left(\hat\vartheta_N - \vartheta\right) = \sqrt{N}\begin{pmatrix} \widehat{H}_N - H \\ \hat\kappa_N^2 - \kappa^2 \\ \hat\sigma_N^2 - \sigma^2 \end{pmatrix} \xrightarrow{\mathrm{d}} \mathcal{N}\left(\vec{0}, \Sigma^0\right)$$

with the asymptotic covariance matrix Σ^0, which can be found by the formula

$$\Sigma^0 = g'(\tau_0)\widetilde{\Sigma}\left(g'(\tau_0)\right)^\top,$$

where $\widetilde{\Sigma}$ is defined in Theorem 2.1 and

$$g'(\tau_0) = \begin{pmatrix} 0 & G_{12} & G_{13} \\ 0 & G_{22} & G_{23} \\ \frac{1}{h} & \frac{4(1-2^{2H})}{h(2^{2H}-2)^2} & \frac{2}{h(2^{2H}-2)^2} \end{pmatrix}$$

[2] The function $f\colon \mathbb{R}^d \to \mathbb{R}$ is said to have Hermite rank equal to q with respect to a random vector X if (a) $\mathsf{E}[(f(X)-\mathsf{E}[f(X)])p_m(X)] = 0$ for every polynomial p_m (on $\mathbb{R}^d$) of degree $m \leqslant q-1$; and (b) there exists a polynomial p_q of degree q such that $\mathsf{E}[(f(X)-\mathsf{E}[f(X)])p_m(X)] \neq 0$, see [10].

with

$$G_{12} = \frac{-1}{2\kappa^2 h^{2H}(2^{2H-1}-1)\log 2},$$
$$G_{13} = \frac{1}{2\kappa^2 h^{2H} 2^{2H}(2^{2H-1}-1)\log 2},$$
$$G_{22} = \frac{2}{h^{2H}} \frac{(2+\log_{2+} h)2^{2H} - 2(\log_{2+} h + 1)}{(2^{2H}-2)^2},$$
$$G_{23} = \frac{2}{h^{2H}} \frac{(2\log_{2+} h - (\log_{2+} h + 1)2^{2H})}{2^{2H}(2^{2H}-2)^2}.$$

Proof. Let us introduce the following function of $\tau = (\xi, \eta, \zeta)^\top$:

$$g(\tau) = \big(g_1(\tau), g_2(\tau), g_3(\tau)\big),$$

where

$$g_1(\tau) = g_1(\xi,\eta,\zeta) = \frac{1}{2}\log_{2+}\frac{\zeta}{\eta},$$
$$g_2(\tau) = g_2(\xi,\eta,\zeta) = \frac{\eta}{h^{2g_1(\xi,\eta,\zeta)}(2^{2g_1(\xi,\eta,\zeta)-1}-1)},$$
$$g_3(\tau) = g_3(\xi,\eta,\zeta) = \frac{1}{h}\left(\xi - g_2(\xi,\eta,\zeta)h^{2g_1(\xi,\eta,\zeta)}\right).$$

Then the estimator (16) can be presented in the following form

$$\hat{\vartheta}_N = (\widehat{H}_N, \hat{\kappa}_N^2, \hat{\sigma}_N^2)^\top = g(\xi_N, \eta_N, \zeta_N).$$

To prove the asymptotic normality of $\hat{\vartheta}_N$ we will apply the delta method to function $g(\tau)$ and sequence τ_N that is asymptotically normal by Theorem 2.1. For this we will need to find matrix $g'(\tau)$ and to show that it is non-singular at τ_0. We will start with evaluating partial derivatives of function g_1. Since

$$g_1(\xi,\eta,\zeta) = \frac{1}{2}\log_{2+}\frac{\zeta}{\eta} = \frac{\log\zeta - \log\eta}{2\log 2},$$

we see that

$$\frac{\partial g_1}{\partial \xi} = 0, \quad \frac{\partial g_1}{\partial \eta} = -\frac{1}{2\eta\log 2}, \quad \frac{\partial g_1}{\partial \zeta} = \frac{1}{2\zeta\log 2}.$$

In order to evaluate partial derivatives of g_2 and g_3, let us consider two intermediate functions:

$$h^{2g_1(\xi,\eta,\zeta)} = h^{\log_{2+}\frac{\zeta}{\eta}} = \left(\frac{\zeta}{\eta}\right)^{\log_{2+} h} = \left(\frac{\zeta}{\eta}\right)^c,$$

where $c = \log_{2+} h$, and

$$2^{2g_1(\xi,\eta,\zeta)-1} - 1 = 2^{\log_{2+}\frac{\zeta}{\eta}-1} - 1 = \frac{\zeta}{2\eta} - 1 = \frac{\zeta - 2\eta}{2\eta}.$$

Then

$$g_2(\xi,\eta,\zeta) = \frac{\eta}{h^{2g_1(\xi,\eta,\zeta)}(2^{2g_1(\xi,\eta,\zeta)-1}-1)} = \frac{\eta}{\left(\frac{\zeta}{\eta}\right)^c \cdot \frac{\zeta-2\eta}{2\eta}} = \frac{2\eta^{2+c}}{\zeta^{c+1} - 2\eta\zeta^c}.$$

Therefore, the corresponding partial derivatives are equal to

$$\frac{\partial g_2}{\partial \xi} = 0,$$
$$\frac{\partial g_2}{\partial \eta} = 2\frac{(2+c)\eta^{1+c}(\zeta^{c+1}-2\eta\zeta^c) + 2\eta^{2+c}\zeta^c}{(\zeta^{c+1}-2\eta\zeta^c)^2} = 2\frac{(2+c)\eta^{1+c}\zeta^{c+1} - 2(c+1)\eta^{2+c}\zeta^c}{(\zeta^{c+1}-2\eta\zeta^c)^2} = 2\left(\frac{\eta}{\zeta}\right)^c \frac{(2+c)\eta\zeta - 2(c+1)\eta^2}{(\zeta-2\eta)^2},$$
$$\frac{\partial g_2}{\partial \zeta} = 2\eta^{2+c}\cdot\frac{-1}{(\zeta^{c+1}-2\eta\zeta^c)^2} \times \big((c+1)\zeta^c - 2c\eta\zeta^{c-1}\big) = 2\left(\frac{\eta}{\zeta}\right)^c \frac{\eta^2(2c\eta - (c+1)\zeta)}{\zeta(\zeta-2\eta)^2}.$$

The function g_3 can be rewritten as

$$g_3(\xi,\eta,\zeta) = \frac{1}{h}\left(\xi - \frac{\eta}{h^{2g_1(\xi,\eta,\zeta)}(2^{2g_1(\xi,\eta,\zeta)-1})} \times h^{2g_1(\xi,\eta,\zeta)}\right) = \frac{1}{h}\left(\xi - \frac{2\eta^2}{\zeta - 2\eta}\right).$$

Hence,

$$\frac{\partial g_3}{\partial \xi} = \frac{1}{h},$$
$$\frac{\partial g_3}{\partial \eta} = \frac{(-1)}{h}\cdot\frac{4\eta(\zeta-2\eta) - 2\eta^2(-2)}{(\zeta-2\eta)^2} = \frac{4\eta(\eta-\zeta)}{h(\zeta-2\eta)^2},$$
$$\frac{\partial g_3}{\partial \zeta} = \frac{2\eta^2}{h(\zeta-2\eta)^2}.$$

So, we have the following derivative matrix:

$$g'(\xi,\eta,\zeta) = \begin{pmatrix} \frac{\partial g_1}{\partial \xi} & \frac{\partial g_1}{\partial \eta} & \frac{\partial g_1}{\partial \zeta} \\ \frac{\partial g_2}{\partial \xi} & \frac{\partial g_2}{\partial \eta} & \frac{\partial g_2}{\partial \zeta} \\ \frac{\partial g_3}{\partial \xi} & \frac{\partial g_3}{\partial \eta} & \frac{\partial g_3}{\partial \zeta} \end{pmatrix} = \begin{pmatrix} 0 & G_{12}^\tau & G_{13}^\tau \\ 0 & G_{22}^\tau & G_{23}^\tau \\ \frac{1}{h} & \frac{4\eta(\eta-\zeta)}{h(\zeta-2\eta)^2} & \frac{2\eta^2}{h(\zeta-2\eta)^2} \end{pmatrix},$$

with

$$G_{12}^{\tau} = \frac{-1}{2\eta \log 2},$$
$$G_{13}^{\tau} = \frac{1}{2\zeta \log 2},$$
$$G_{22}^{\tau} = 2(\frac{\eta}{\zeta})^c \frac{(2+c)\eta\zeta - 2(c+1)\eta^2}{(\zeta - 2\eta)^2},$$
$$G_{23}^{\tau} = 2(\frac{\eta}{\zeta})^c \frac{\eta^2(2c\eta - (c+1)\zeta)}{\zeta(\zeta - 2\eta)^2},$$

where $c = \log_{2^+} h$.

At $\tau = \tau_0$

$$g'(\tau_0) = \begin{pmatrix} 0 & G_{12} & G_{13} \\ 0 & G_{22} & G_{23} \\ \frac{1}{h} & \frac{4(1-2^{2H})}{h(2^{2H}-2)^2} & \frac{2}{h(2^{2H}-2)^2} \end{pmatrix},$$

where G_{12}, G_{13}, G_{22}, G_{23} are defined in the in the formulation of the theorem.

It is not hard to see that the determinant of this matrix

$$\det(g'(\tau_0)) = -\frac{1}{2^{2H+1}\kappa^2 h^{4H+1}(2^{2H-1}-1)^2 \log 2}.$$

is well defined for $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{4})$ and it is not equal to zero. Therefore the matrix of derivatives is non-singular at $\tau = \tau_0$ and so the delta method can be applied, see, e. g., [11, Theorem B.6]. This method implies the statement of the theorem and provides with the formula for asymptotic covariance matrix Σ^0. □

4 Simulation study

In this section we would like to present results of estimators performance by numerical simulations. For each generated trajectory we will estimate asymptotic covariance matrices defined in (2.1) and (3.1) by using values of estimators (4). For each set of parameters we generate 1000 trajectories of the process X_t and calculate the empirical means, empirical standard deviations of the estimates, the square root of estimated asymptotic variation divided by N ($\sqrt{\hat{\sigma}/N}$) and cover probability (CP) for $\alpha = 5\%$ based on estimator of asymptotic covariance matrix. Series alike (7) are divided into sum of two convergent series

$$\sum_{i=-\infty}^{+\infty} \tilde{\rho}(i+\alpha)\tilde{\rho}(i+\beta) = \sum_{i=0}^{+\infty} \tilde{\rho}(i+\alpha)\tilde{\rho}(i+\beta) + \sum_{i=0}^{+\infty} \tilde{\rho}(i+1-\alpha)\tilde{\rho}(i+1-\beta),$$

where $\alpha, \beta \in \mathbb{Z}$. Series $\sum_{i=0}^{+\infty} \tilde{\rho}(i+\alpha)\tilde{\rho}(i+\beta)$ are estimated with precision $\delta = 10^{-3}$ by evaluating the sum $S_m = \sum_{i=0}^{m} \tilde{\rho}(i+\alpha)\tilde{\rho}(i+\beta)$ with the first m such that $|S_m - S_{m-1}| < 10^{-3}$. Also, for simulation purposes, we will be using only $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{4})$ because otherwise the series (7) will not converge thus the asymptotic covariance matrix will be non-evaluable. Moreover, the same will be applied for each case when the estimated $\widehat{H}_N$ will be out of boundaries $(0, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{4})$.

Table 1: The estimator $\widehat{H}_N$ with $\kappa^2 = 6.25$, $\sigma^2 = 2.25$ ($h = 1$)

H		N: 2^8	2^{12}	2^{16}	2^{20}
0.2	Mean	0.1823	0.2018	0.2008	0.2000
	S.dev.	0.3135	0.0721	0.0176	0.0043
	$\sqrt{\hat{\sigma}/N}$	0.3027	0.0728	0.0181	0.0045
	CP%	100.00	97.522	95.600	96.400
0.4	Mean	0.4281	0.4052	0.4009	0.3994
	S.dev.	0.7990	0.1686	0.0403	0.0103
	$\sqrt{\hat{\sigma}/N}$	0.6739	0.1715	0.0404	0.0100
	CP%	100.00	99.636	97.005	94.795
0.6	Mean	0.5963	0.5903	0.5999	0.5999
	S.dev.	0.5792	0.1232	0.0290	0.0070
	$\sqrt{\hat{\sigma}/N}$	0.4746	0.1119	0.0279	0.0070
	CP%	100.00	100.00	95.755	95.496

Table 2: The estimator $\hat{\kappa}_N^2$ with $\kappa^2 = 6.25$, $\sigma^2 = 2.25$ ($h = 1$)

H		N: 2^8	2^{12}	2^{16}	2^{20}
0.2	Mean	13.197	6.5692	6.2721	6.2505
	S.dev.	86.074	1.3784	0.2560	0.0604
	$\sqrt{\hat{\sigma}/N}$	4.0819	1.0934	0.2569	0.0633
	CP%	100.00	93.427	95.500	95.900
0.4	Mean	8.5306	25.768	8.0469	6.2735
	S.dev.	54.206	206.04	9.9068	0.5839
	$\sqrt{\hat{\sigma}/N}$	13.130	4.2116	2.0175	0.5636
	CP%	98.131	70.909	88.412	94.094
0.6	Mean	9.1450	26.615	10.454	6.2924
	S.dev.	39.869	269.74	104.39	0.5002
	$\sqrt{\hat{\sigma}/N}$	17.947	4.5897	1.7836	0.4920
	CP%	100.00	83.200	88.639	94.795

Table 3: The estimator $\hat{\sigma}_N^2$ with $\kappa^2 = 6.25$, $\sigma^2 = 2.25$ ($h = 1$)

H		N: 2^8	2^{12}	2^{16}	2^{20}
0.2	Mean	3.2413	2.1907	2.2265	2.2492
	S.dev.	1.4481	0.8823	0.2542	0.0602
	$\sqrt{\hat{\sigma}/N}$	4.0495	1.0873	0.2554	0.0629
	CP%	100.00	92.996	95.900	96.400
0.4	Mean	5.7222	4.4137	2.6526	2.2297
	S.dev.	1.8376	1.6134	1.2686	0.5759
	$\sqrt{\hat{\sigma}/N}$	13.129	4.2119	2.0175	0.5636
	CP%	99.533	70.000	89.063	94.094
0.6	Mean	6.2802	4.5294	2.6477	2.2096
	S.dev.	2.2077	1.8375	1.2471	0.4946
	$\sqrt{\hat{\sigma}/N}$	17.972	4.5923	1.7840	0.4920
	CP%	100.00	82.600	88.764	94.795

In Tables 1–3 one can find performance of the estimators $\widehat{H}_N$, $\hat{\kappa}_N^2$, and $\hat{\sigma}_N^2$, for the following true values of the parameters: $\kappa = 2.5$, $\sigma = 1.5$. We observe that the cover probability and asymptotic variance estimators demonstrate better results for H in (0, 1/2) in comparison with H in (1/2, 3/4). Moreover, the closer parameter H to 1/2 the lower rate of convergence for constructed estimators and this is consistent with numerical simulations results obtained in [6].

We believe that partially observed results are affected by convergence rates of series defined in Lemma 1. These series do converge for $H \in (\frac{1}{2}, \frac{3}{4})$ but with a such significantly slow rate that it actually has an impact on estimating the sum of the corresponding series. At the same time for $H \in (0, \frac{1}{2})$ there is no significant impact. Moreover, the convergence rates for $H \in (\frac{1}{2}, \frac{3}{4})$ creates some technical difficulties. As was previously stated we evaluate such series by splitting them in two. We have conducted simulations for the following series:

$$\sum_{i=0}^{+\infty} \tilde{\rho}(i)^2. \tag{17}$$

We estimate (17) by finding the first $m \in \mathbb{N}$ such that for $S_m = \sum_{i=0}^{m} \tilde{\rho}(i)^2$ for some pre-defined δ it holds that $|S_m - S_{m-1}| \leqslant \delta$.

This estimation was conducted for $\delta \in \{10^{-3}, 10^{-5}, 10^{-7}, 10^{-9}\}$ for $\kappa = 2.5$, $\sigma = 1.5$ and $H \in \{0.2, 0.6\}$. Estimation results are presented in Table 4. As one can see, in order to estimate S_m with a precision of $\delta = 10^{-10}$ for $H = 0.2$ only a few hundred iterations are needed, while for $H = 0.6$ it requires hundreds of thousands of iterations. Moreover, lowering the precision from 10^{-10} to 10^{-3} has almost no impact on the estimated value S_m for low H, but for high H it could result in a difference around 1%.

Table 4: The $\widetilde{\Sigma}_{11}$ convergence with $\kappa^2 = 6.25$, $\sigma^2 = 2.25$ ($h = 1$)

H		δ: 10^{-3}	10^{-5}	2^{-7}	10^{-9}
0.2	m	8	31	129	543
	S_m	76.879	76.880	76.881	76.881
0.6	m	53	931	16552	$2.9 * 10^5$
	S_m	73,768	73.831	73.851	73.853

Additionally, we want to note that for a small number N there is a high percentage of iterations that result in some estimators (4) being non-evaluable (Non Ev.%) or estimated value of H being out of interval $(0, \frac{1}{2}) \cap (\frac{1}{2}, \frac{3}{4})$ (H Out%). In such cases, asymptotic covariance matrices cannot be estimated because the required series (7) will not converge. Corresponding simulation results can be found in Table 5.

Table 5: Percentage of estimated results that cannot be used for estimating asymptotic covariance matrix in previous simulations

H		N: 2^8	2^{12}	2^{16}	2^{20}
0.2	Non Ev.%	34.10	6.900	0.000	0.000
	H Out%	25.80	0.300	0.000	0.000
0.4	Non Ev.%	59.00	44.10	23.20	0.100
	H Out%	19.60	0.900	0.000	0.000
0.6	Non Ev.%	52.60	42.20	19.90	0.100
	H Out%	31.80	7.800	0.000	0.000

5 Conclusion

Joint asymptotic normality of estimators for parameters (H, κ, σ) constructed in [6] is proved for $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{4})$ and respective asymptotoc covariance matrix is estimated. Provided simulations show that constructed estimators are much more accurate for low values of H, while for high values of H they are less accurate and much harder to compute. In general, the obtained results are consistent with conclusions from [6].

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