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Automorphisms of Mackey groups

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Розглядаються тотальні підпростори в просторі лінійних функціоналів над нескінченно-вимірними векторними просторами та пов'язані з ними алгебри і групи Маккі. Ця стаття присвячена опису автоморфізмів груп Маккі $SL_\infty(V|W)$, $O_\infty(f)$ та $SU_\infty(f)$ над полями характеристики, відмінної від 2 або 3. Для цього досліджується зв'язок між автоморфізмами полів та автоморфізмами згаданих груп.

Дж.Холл довів, що нескінченними простими фінітарними періодичними групами є або знакозмінні групи на нескінченних множинах, або групи Маккі над полями, що є алгебричними розширеннями скінченних полів. Автоморфізми нескінченних знакозмінних груп були описані в роботах Дж.Шрайера та С.Улама. Описом автоморфізмів фінітарних груп Маккі та спеціальних фінітарних унітарних груп Маккі ми завершуємо класифікацію автоморфізмів усіх нескінченних простих фінітарних періодичних груп над полями характеристики, що не дорівнює 2 або 3. Доведення ґрунтується на описі автоморфізмів елементарних лінійних груп над асоціативними кільцями, який належить І.Голубчику, О.Михальову та Ю.Зельманову.

Ключові слова: Автоморфізм, анти-автоморфізм, нескінченна проста фінітарна періодична група, група Маккі.

We consider total subspaces of linear functionals on an infinite-dimensional vector space and the related Mackey algebras and groups. We outline the description of automorphisms of Mackey groups $SL_\infty(V|W)$, $O_\infty(f)$, and $SU_\infty(f)$ over fields of characteristics not equal to 2, 3. Moreover, the paper explores the relationship between field automorphisms and automorphisms of the aforementioned groups.

J.Hall proved that infinite simple finitary torsion groups are the alternating groups on infinite sets or Mackey groups over a field, which is an algebraic extension of a finite field. J.Schreier and S.Ulam described automorphisms of infinite alternating groups. With the description of automorphisms of finitary Mackey groups and special finitary unitary Mackey groups we finish classification of automorphisms of all infinite simple finitary torsion groups over fields of characteristics not equal to 2, 3. The proof is based of description of automorphisms of elementary linear groups over associative rings that due to I.Golubchik, A.Mikhalev and E.Zelmanov.

Key Words: Automorphisms, anti-automorphisms, infinite simple finatary torsion group, Mackey group.

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1 Introduction

Let $\mathbb{F}$ be a field of characteristic $\neq 2, 3$; and let V be an infinite-dimensional vector space over $\mathbb{F}$. Let V^* denote the vector space of all linear functionals $V \to \mathbb{F}$. For an element $v \in V$ and a linear functional $w \in V^*$, we denote $w(v) = (v|w)$.

A subspace $W \subset V^*$ is called *total* if $v \in V$, $(v|W) = (0)$ implies $v = 0$.

Let $\mathrm{End}_{\mathbb{F}}(V)$ be the associative algebra of all linear transformations $V \to V$. Additionally, define

$$\mathrm{End}_{fin}(V) = \{\varphi \in \mathrm{End}_{\mathbb{F}}(V) \,\big|\, \dim_{\mathbb{F}} \varphi(V) < \infty\}.$$

The space V^* has a natural structure of a right $\mathrm{End}_{\mathbb{F}}(V)$-module. Given a linear functional $\chi : V \to \mathbb{F}$ and a linear transformation $\varphi : V \to V$ we define

$$(\chi\varphi)(v) = \chi(\varphi(v)), \quad v \in V.$$

The subalgebras

$$A(V|W) = \{\varphi \in \mathrm{End}_{\mathbb{F}}(V) \,\big|\, W\varphi \subseteq W\}$$

and

$$A_{fin}(V|W) = A(V|W) \cap \mathrm{End}_{fin}(V|W)$$

are called the *Mackey algebra* and the *finitary Mackey algebra*, respectively. Clearly, $A_{fin}(V|W)$ is an ideal of the algebra $A(V|W)$. For more information about Mackey algebra, see [6, 7, 8].

The algebra $A_{fin}(V|W)$ can be identified with the tensor product

$$V \otimes_{\mathbb{F}} W, \quad (v_1 \otimes w_1)(v_2 \otimes w_2) = (v_2|w_1)v_1 \otimes w_2.$$

Here, an element $\sum_i v_i \otimes w_i$ is viewed as the linear transformation

$$v \to \sum_i (v|w_i)v_i, \quad v \in V.$$

The algebra $A_{fin}(V|W)$ of a total subspace $W \subset V^*$ is a nonunital locally matrix algebra; see [1, 3, 4].

Let $GL(V)$ denote the group of all invertible linear transformations on V. The Mackey algebras $A(V|W)$ and $A_{fin}(V|W)$ give rise to the groups:

$$GL(V|W) = A(V|W) \cap GL(V);$$

$$GL_\infty(V|W) = \big(A_{fin}(V|W) + Id_V\big) \cap GL(V),$$

where Id_V is the identity mapping $V \to V$. We also consider the group

$$SL_\infty(V|W) = [GL_\infty(V|W), GL_\infty(V|W)].$$

This group $SL_\infty(V|W)$ is a direct limit of groups $SL_n(\mathbb{F})$, and it is a diagonally local SL-group in the sense of [2].

A bijective linear transformation (respectively, additive map) φ of an associative algebra (respectively, ring) A is called an *anti-automorphism* if

$$\varphi(ab) = \varphi(b)\varphi(a)$$

for arbitrary elements $a, b \in A$. An anti-automorphism $*$ is called an *involution* if

$$(a^*)^* = a \quad \text{for an arbitrary element} \quad a \in A.$$

N. Jacobson [6] showed that a finitary Mackey algebra $A_{fin}(V|W)$ has an involution if and only if the vector space V is equipped with a weakly Hermition nondegenerate bilinear form $f(x, y)$ such that W constitutes the space of linear functionals

$$\widehat{x} : v \to f(v, x), \quad v \in V,$$

where x runs over V. In this case, we call the pair (V, W) *dual*; and the anti-automorphism is the transpose

$$\varphi \to \varphi^t, \quad f(\varphi(v_1), v_2) = f(v_1, \varphi^t(v_2)).$$

Let us note that a linear transformation $\varphi \in \mathrm{End}_{\mathbb{F}}(M)$ has a transpose if and only if it lies in $A(V|W)$.

We also note that an anti-automorphism t is an involution if and only if the form f is symmetric or skew-symmetric.

Consider the group $O(f)$ of orthogonal linear transformations defined as

$$O(f) = \{a \in A(V|W) \,\big|\, a^t = a^{-1}\}$$

and

$$O_\infty(f) = O(f) \cap GL_\infty(V|W).$$

If the field $\mathbb{F}$ is equipped with an automorphism of order 2: $\alpha \to \bar{\alpha}$, $\alpha \in \mathbb{F}$, and the form f is semilinear, i.e.

$$f(\alpha v_1, v_2) = \alpha f(v_1, v_2);\ f(v_1, \alpha v_2) = \bar{\alpha} f(v_1, v_2),$$

then we define unitaly groups as follows:

$$U(f) = \{a \in A(V|W) \,\big|\, \bar{a}^t = \bar{a}^{-1}\},$$

$$U_\infty(f) = U(f) \cap GL_\infty(V|W)$$

and their special analogs:

$$SU(f) = [U(f), U(f)],$$

$$SU_\infty(f) = [U_\infty(f), U_\infty(f)].$$

We also consider scalar-unitary groups defined as follows:

$$\tilde{U}(f) = \{a \in A(V|W) \,\big|\, \bar{a}^t = \alpha a^{-1}, 0 \neq \alpha \in \mathbb{F}\}.$$

J. Hall [5] classified infinite simple finitary torsion groups. He showed that a group belongs to this class if and only if it is isomorphic to an alternating group $\mathrm{Alt}(X)$ on an inifinite set X or to one of Mackey groups $SL_\infty(V|W)$, $O_\infty(f)$, $SU_\infty(\mathbb{F})$, where the ground field $\mathbb{F}$ is locally finite, and f is a nondegenerate (skew) symmetric (semi)linear form. The proof uses the Classification of Finite Simple Groups.

2 Main results

The following two theorems describe automorphisms of Mackey groups

$$SL_\infty(V|W), \quad O_\infty(f), \quad SU_\infty(f).$$

In particular, they describe automorphisms of all infinite simple finitary torsion groups in J. Hall's classification. Recall that automorphisms of infinite alternating groups were described by J. Schrerier and S. Ulam [9].

Theorem 1. (a) *Let the characteristic of the field $\mathbb{F}$ is not equal to 2 or 3. Then an arbitrary automorphism φ of the group $SL_\infty(V|W)$ either extends to a ring automorphism of $A_{fin}(V|W)$ or, if the spaces V and W are dual, there exists an anti-automorphism ψ of $A_{fin}(V|W)$ such that*

$$\varphi(g) = \psi\big((g^{-1})^t\big)$$

for an arbitrary element $g \in SL_\infty(V|W)$.

(b) *Let the characteristic of the field $\mathbb{F}$ is not equal to 2 or 3. Then an arbitrary automorphism of the group $O_\infty(f)$ or $SU_\infty(f)$ extends to a ring automorphism of $A_{fin}(V|W)$.*

N. Jacobson [6] showed that all $\mathbb{F}$-linear automorphisms of $A_{fin}(V|W)$ are conjugations by elements from $GL(V|W)$. Hence,

$$\mathrm{Aut}_{\mathbb{F}}\big(A_{fin}(V|W)\big) \cong$$

$$\cong GL(V|W)\big/\mathrm{Center} = PGL(V|W).$$

A ring automorphism of $A_{fin}(V|W)$ is semilinear with respect to some automorphism $\sigma \in \mathrm{Aut}(\mathbb{F})$ of the field $\mathbb{F}$.

Let the subgroup $\mathrm{Aut}_{V,W}(\mathbb{F}) \leqslant \mathrm{Aut}(\mathbb{F})$ consist of those automorphisms $\sigma \in \mathrm{Aut}(\mathbb{F})$ that extend to σ-semilinear automorphisms of $A_{fin}(V|W)$.

If both spaces V and W are countable-dimensional, then

$$\mathrm{Aut}_{V,W}(\mathbb{F}) = \mathrm{Aut}(\mathbb{F}).$$

If the spaces V and W have uncountable dimensions, then the subgroup $\mathrm{Aut}_{V,W}(\mathbb{F})$ may be strictly smaller than $\mathrm{Aut}(\mathbb{F})$.

Theorem 2. *If the spaces V and W are not dual, then the group $\mathrm{Aut}\big(SL_\infty(V|W)\big)$ is an extension of $\mathrm{Aut}_{V,W}(\mathbb{F})$ by $PGL(V|W)$. If the spaces V and W are dual, then*

$$\mathrm{Aut}\big(SL_\infty(V|W)\big)\big/\mathrm{Aut}_{\mathbb{F}}\big(A_{fin}(V|W)\big) \cong$$

$$\cong \mathrm{Aut}_{V,W}(\mathbb{F}),$$

$$\mathrm{Aut}_{\mathbb{F}}\big(A_{fin}(V|W)\big)\big/PGL(V|W) \cong \mathbb{Z}/2\mathbb{Z}.$$

The proof of Theorems 1, 2 are based on the description of automorphisms of elementary linear groups over associative rings due to I. Golubchik, A. Mikhalev and E. Zelmanov; and on the fact that $A_{fin}(V|W)$ is locally covered by algebras, to the automorphisms or anti-automorphisms of which the automorphisms of elementary matrix groups are lifted.

The question about automorphisms of Mackey groups in characteristics 2, 3 remains open.

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