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Асимптотика м'якого розв'язку параболічного рівняння із загальною стохастичною мірою

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Asymptotics of the mild solution of a parabolic equation with a general stochastic measure

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Анотація. Розглянуто задачу Коші для параболічного рівняння в одновимірному просторі, керованого загальною стохастичною мірою. Доведено, що за певних умов м'який розв'язок прямує до нуля майже напевно, якщо абсолютна величина просторової змінної прямує до нескінченності.

Ключові слова: асимптотична поведінка, стохастична міра, стохастичне параболічне рівняння, м'який розв'язок, простір Бесова.

Abstract. We study the Cauchy problem for a parabolic equation on the line driven by a general stochastic measure. Under some assumptions, we prove that the mild solution tends to zero almost surely as the absolute value of the spatial variable tends to infinity.

Key Words: asymptotic behavior, stochastic measure, stochastic parabolic equation, mild solution, Besov space.

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1 Introduction

We continue investigation of the parabolic equation driven by a general stochastic measure. Namely, we study the asymptotic behavior of the mild solution.

Let $\mathsf{L}_0(\Omega, \mathcal{F}, \mathsf{P})$ be the set of all real-valued random variables defined on complete probability space $(\Omega, \mathcal{F}, \mathsf{P})$, X be a metric space and $\mathcal{B}(\mathsf{X})$ be a σ-algebra of Borel subsets of X. Let μ be a general stochastic measure (or simply «stochastic measure») on $\mathcal{B}(\mathsf{X})$, i.e. a σ-additive mapping $\mu : \mathcal{B}(\mathsf{X}) \to \mathsf{L}_0(\Omega, \mathcal{F}, \mathsf{P})$ (see, for example, [1, Section 7]). We do not require the fulfillment of any other assumptions except σ-additivity.

Consider the following equation

$$\begin{cases} Lu(t,x)dt + f(t,x,u(t,x))dt + \sigma(t,x)d\mu(x) = 0, \\ u(0,x) = u_0(x), \end{cases} \quad (1)$$

where $(t,x) \in [0,T] \times \mathbb{R}$, μ is a general stochastic measure defined on the σ-algebra of Borel subsets of $\mathbb{R}$, and the operator L has the form

$$Lu(t,x) = a(t)\frac{\partial^2 u(t,x)}{\partial^2 x} + b(t)\frac{\partial u(t,x)}{\partial x} + c(t)u(t,x) - \frac{\partial u(t,x)}{\partial t}, \quad (2)$$

where functions a, b, c are defined on the set

$$S = [0,T] \times \mathbb{R} = \{(t,x) : t \in [0,T],\ x \in \mathbb{R}\}.$$

The solution of equation (1) is understood in the following mild sense

$$u(t,x) = \int_{\mathbb{R}} p(t,x;0,y)u_0(y)\,dy + \int_0^t ds \int_{\mathbb{R}} p(t,x;s,y) f(s,y,u(s,y))\,dy + \int_{\mathbb{R}} d\mu(y) \int_0^t p(t,x;s,y)\sigma(s,y)\,ds\,. \quad (3)$$

Here $p(t,x;s,y)$ is the fundamental solution of the operator L defined by (2). The integrals of random functions with respect to ds and dy are taken for each fixed ω. Some approaches to a definition of such integrals and their properties are investigated in [2]. For more details about the definition of the mild solution see [3, Chapter 6].

The existence, uniqueness and Hölder regularity of the solution of equation (3) are proved in [4].

Our aim here is to establish the conditions under which that the solution of problem (3) tends to zero almost surely as the absolute value of the spatial variable tends to infinity. The same result is presented in [5] for the heat equation. We generalize it to the case of a parabolic equation.

Boris Manikin in [6] obtained the asymptotic properties of the same parabolic equation as $t \to \infty$. The particular case of this problem (heat equation) is considered in [7].

Similar asymptotic behavior and the regularity of the mild solution of the wave equation with general stochastic measure in three-dimensional space are investigated in [8] and [9].

The heat equation with stochastic measure depending on time variable t, is considered in [10], where it was proved that the mild solution tends to zero as $|x| \to \infty$.

2 Preliminaries

Consider the Besov space $B^{\alpha}_{22}([b,c])$, $\alpha \in (1/2,1)$, $b,c \in \mathbb{R}$, that is, the space of functions $g:[b,c] \to \mathbb{R}$ such that the norm

$$\|g\|_{B^{\alpha}_{22}([b,c])} = \|g\|_{\mathsf{L}_2([b,c])} + \\ + \Big(\int_0^{c-b} \big(w_{2,[b,c]}(g,r)\big)^2 r^{-2\alpha-1}\, dr\Big)^{\frac{1}{2}},$$

is finite. Here

$$w_{2,[b,c]}(g,r) = \sup_{0\leqslant h\leqslant r} \left(\int_b^{c-h} |g(s+h)-g(s)|^2\, ds\right)^{\frac{1}{2}}.$$

For any fixed $j \in \mathbb{Z}$ put

$$\Delta^{(j)}_{kn} = (j+(k-1)2^{-n}, j+k2^{-n}], \\ n \geqslant 0, \quad 1 \leqslant k \leqslant 2^n.$$

Let function

$$g(t,x,v) : [0,T]\times\mathbb{R}\times[j,j+1] \to \mathbb{R}$$

be such that $\forall (t,x) \in [0,T]\times\mathbb{R}:\ g(t,x,\cdot)$ is continuous on $[j,j+1]$.

Put

$$g_n(t,x,v) = g(t,x,j)\mathbb{1}_{\{j\}}(v) + \\ + \sum_{1\leqslant k\leqslant 2^n} g(t,x,j+(k-1)2^{-n})\mathbb{1}_{\Delta^{(j)}_{kn}}(v).$$

Then by [11, Lemma 3.2] the stochastic function

$$\eta_j(t,x) = \int_{[j,j+1]} g(t,x,v)d\mu(v),\ (t,x) \in [0,T]\times\mathbb{R},$$

has the version

$$\widetilde{\eta}_j(t,x) = \int_{[j,j+1]} g_0(t,x,v)d\mu(v) + \\ + \sum_{n\geqslant 1}\int_{[j,j+1]} \big(g_n(t,x,v) - g_{n-1}(t,x,v)\big)d\mu(v), \tag{4}$$

such that, for all

$$\varepsilon > 0,\ \omega \in \Omega, (t,x) \in [0,T]\times\mathbb{R},$$

we get

$$|\widetilde{\eta}_j(t,x)| \leqslant \big|g(t,x,j)\mu\big([j,j+1]\big)\big| + \\ + \left\{\sum_{n\geqslant 1} 2^{n\varepsilon} \sum_{1\leqslant k\leqslant 2^n} \big|g(t,x,j+k2^{-n}) - \\ - g(t,x,j+(k-1)2^{-n})\big|^2\right\}^{\frac{1}{2}} \times \\ \times \left\{\sum_{n\geqslant 1} 2^{-n\varepsilon} \sum_{1\leqslant k\leqslant 2^n} \Big|\mu\Big(\Delta^{(j)}_{kn}\Big)\Big|^2\right\}^{\frac{1}{2}}.$$

For this version we have, by [12, Theorem 1.2],

$$|\widetilde{\eta}_j(t,x)| \leqslant \\ \leqslant |g(t,x,j)\mu([j,j+1])| + C\|g(t,x,\cdot)\|_{B^{\alpha}_{22}([j,j+1])} \times \\ \times \Big\{\sum_{n\geqslant 1} 2^{-n\varepsilon} \sum_{1\leqslant k\leqslant 2^n} |\mu(\Delta^{(j)}_{kn})|^2\Big\}^{\frac{1}{2}}, \tag{5}$$

where $\alpha = \varepsilon/2+1/2$ and the sum with a stochastic measure is finite by [11, Lemma 3.1]. Note that we can chose the same version (4) for all $(t,x) \in [0,T]\times\mathbb{R}$, while the constant C depends on α and does not depend on j,t,x,ω.

3 Main result

The following conditions are assumed throughout the paper.

A1. Function $u_0(y) = u_0(y,\omega) : \mathbb{R}\times\Omega \to \mathbb{R}$ is measurable and bounded, namely,

$$|u_0(y,\omega)| \leqslant C_{u_0}(\omega).$$

A2. Function $u_0(y)$ is Hölder continuous in $y \in \mathbb{R}$, that is,

$$|u_0(y_1) - u_0(y_2)| \leqslant L_{u_0}(\omega)\,|y_1 - y_2|^{\beta(u_0)}\,,\\ \beta(u_0) \geqslant 1/2\,.$$

A3. Function $f(s,y,z) : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is measurable and bounded, namely,

$$|f(s,y,z)| \leqslant C_f.$$

A4. Function $f(s,y,z)$ is uniformly Lipschitz in $y \in \mathbb{R},\ z \in \mathbb{R}$, that is,

$$|f(s,y_1,z_1) - f(s,y_2,z_2)| \leqslant L_f(|y_1 - y_2| + |z_1 - z_2|).$$

A5. Function $\sigma(s,y) : [0,T] \times \mathbb{R} \to \mathbb{R}$ is measurable and bounded, namely,

$$|\sigma(s,y)| \leqslant C_\sigma.$$

A6. Function $\sigma(s,y)$ is Hölder continuous in $y \in \mathbb{R}$, namely,

$$|\sigma(s,y_1) - \sigma(s,y_2)| \leqslant L_\sigma\,|y_1 - y_2|^{\beta(\sigma)},\ \beta(\sigma) > 1/2.$$

A7. Fundamental solution p of the operator L is homogeneous in the space variable, namely,

$$p(t,x;s,y) = p(t,x-y;s).$$

A8. Functions $a(t),\ b(t),\ c(t)$ are continuous in $[0,T]$ and there are $\alpha > 0,\ A > 0, A_0 > 0$ such that

$$|a(t_1) - a(t_2)| \leqslant A\,|t_1 - t_2|^{\alpha}\,,\ a(t_1) \geqslant A_0,$$

for all $t_1, t_2 \in [0,T]$.

A9. Function $|y|^\tau$ is integrable with respect to μ on $\mathbb{R}$ for some $\tau > 1/2$.

A10. $|u_0(y)| \to 0,\ \sup\limits_{s\in[0,T],\, z\in\mathbb{R}} |f(s,y,z)| \to 0$, as $|y| \to \infty$.

Note that we consider the fundamental solution p of the operator L that is homogeneous in the space variable (Assumption A7). In this case coefficients a, b, c do not depend on the space variable.

If Assumption A8 holds, then according to [13, Section 4, Theorem 1] we have

$$|p(t,x;s)| \leqslant M(t-s)^{-\frac{1}{2}} e^{-\frac{\lambda|x|^2}{t-s}}, \tag{6}$$

$$\left|\frac{\partial p(t,x;s)}{\partial x}\right| \leqslant M(t-s)^{-1} e^{-\frac{\lambda|x|^2}{t-s}}, \tag{7}$$

where λ and M are positive constants.

The symbols C and $C(\omega)$ denote nonrandom and random constants which can vary from line to line.

Theorem 3.1. *Suppose Assumptions* A1 – A10 *hold. Then the solution of equation* (3) *has a version $u(t,x)$ such that for all $t \in [0,T]$, $\omega \in \Omega$,*

$$|u(t,x)| \to 0, \quad |x| \to \infty.$$

Proof. According to [4, Theorem], under Assumptions A1 – A7 for all $t \in [0,T]$, $x \in \mathbb{R}$, equation (3) has the unique solution $u(t,x)$.

We have

$$|u(t,x)| \leqslant \left|\int_{\mathbb{R}} p(t,x-y;0)u_0(y)\,dy\right| + \\ + \left|\int_0^t ds \int_{\mathbb{R}} p(t,x-y;s) f(s,y,u(s,y))\,dy\right| + \\ + \left|\int_{\mathbb{R}} d\mu(y) \int_0^t p(t,x-y;s)\sigma(s,y)\,ds\right| = \\ = I_1(t,x) + I_2(t,x) + I_3(t,x). \tag{8}$$

Now each of terms is treated separately. First consider the integral with respect to stochastic measure. Put

$$h(t,x,y) = \int_0^t p(t,x-y;s)\sigma(s,y)\,ds\,,$$

then

$$I_3(t,x) \leqslant \sum_{j\in\mathbb{Z}} \left|\int_{(j,j+1]} h(t,x,y)d\mu(y)\right|.$$

Let arbitrary $j \in \mathbb{Z},\ t \in [0,T],\ x \in \mathbb{R}$, be fixed. Then the random function

$$\eta_j(t,x) = \int_{(j,j+1]} h(t,x,y)d\mu(y)$$

has a version (4) that admits estimate (5). This version is denoted by the same symbol η. Thus, for any $\alpha \in (1/2,1)$, we have

$$|\eta_j(t,x)| \leqslant |h(t,x,j)\mu([j,j+1])| + \\ + C\|h(t,x,\cdot)\|_{B^\alpha_{22}([j,j+1])} \times \\ \times \Big\{\sum_{n\geqslant 1} 2^{-n\varepsilon} \sum_{1\leqslant k\leqslant 2^n} |\mu(\Delta^{(j)}_{kn})|^2\Big\}^{\frac{1}{2}},$$

and

$$I_3(t,x) \leqslant \sum_{j\in\mathbb{Z}} |\eta_j(t,x)| \leqslant J_1(t,x) + J_2(t,x), \tag{9}$$

where

$$J_1(t,x) = \sum_{j\in\mathbb{Z}} |h(t,x,j)\mu([j,j+1])|,$$
$$J_2(t,x) = C\sum_{j\in\mathbb{Z}} \|h(t,x,\cdot)\|_{B^{\alpha}_{22}([j,j+1])}\times$$
$$\times\Big\{\sum_{n\geqslant 1} 2^{-n\varepsilon} \sum_{1\leqslant k\leqslant 2^n} |\mu(\Delta^{(j)}_{kn})|^2\Big\}^{\frac{1}{2}}.$$

Consider the first term from (9). Get

$$J_1(t,x) \leqslant \left(\sum_{j\in\mathbb{Z}} (1+|j|)^{-2\tau}|h(t,x,j)|^2\right)^{\frac{1}{2}}\times$$
$$\times\left(\sum_{j\in\mathbb{Z}} (1+|j|)^{2\tau}|\mu([j,j+1])|^2\right)^{\frac{1}{2}} = F_1\cdot F_2.$$

By Assumption A8 for $\tau > 1/2$ according to [11, Lemma 3.1] for function $(1+|j|)^{\tau}\mathbb{1}_{(j,j+1]}(y)$ we obtain that $F_2 < +\infty$ a.s. And we can assume $u(t,x) = 0$ on the set of ω such that $F_2 = +\infty$.

Further, by Assumptions A5,A7 and upper bound (6) we have

$$|h(t,x,j)| \leqslant C_\sigma \int_0^t |p(t,x-j;s)|\,ds \leqslant$$
$$\leqslant C_\sigma M\int_0^t (t-s)^{-\frac{1}{2}} e^{-\frac{\lambda|x-j|^2}{t-s}}\,ds \leqslant$$
$$\leqslant 2C_\sigma M\sqrt{t} \leqslant C. \qquad (10)$$

Due to the fact $\sum_{j\in\mathbb{Z}}(1+|j|)^{-2\tau} < +\infty$, we conclude that F_1 is bounded. Moreover, by estimates from (10) we obtain, for any fixed $j\in\mathbb{Z}$,

$$h(t,x,j)\to 0,\quad |x|\to\infty. \qquad (11)$$

Thus by the Lebesgue's dominated convergence theorem we get

$$F_1\to 0,\quad |x|\to\infty.$$

Consequently,

$$J_1(t,x)\to 0,\quad |x|\to\infty.$$

Now we consider term $J_2(t,x)$. For it, we have the following result.

$$J_2(t,x) \leqslant C\sum_{j\in\mathbb{Z}} \Bigg[\|h(t,x,\cdot)\|_{\mathsf{L}_2([j,j+1])} +$$
$$+\Big(\int_0^1 \big(w_{2,[j,j+1]}(h,r)\big)^2 r^{-2\alpha-1}\,dr\Big)^{\frac{1}{2}}\Bigg]\times$$
$$\times\Big\{\sum_{n\geqslant 1} 2^{-n\varepsilon} \sum_{1\leqslant k\leqslant 2^n} |\mu(\Delta^{(j)}_{kn})|^2\Big\}^{\frac{1}{2}}$$
$$= C\sum_{j\in\mathbb{Z}} \|h(t,x,\cdot)\|_{\mathsf{L}_2([j,j+1])}\times$$
$$\times\Big\{\sum_{n\geqslant 1} 2^{-n\varepsilon} \sum_{1\leqslant k\leqslant 2^n} |\mu(\Delta^{(j)}_{kn})|^2\Big\}^{\frac{1}{2}} +$$
$$+C\sum_{j\in\mathbb{Z}}\Big(\int_0^1 \big(w_{2,[j,j+1]}(h,r)\big)^2 r^{-2\alpha-1}\,dr\Big)^{\frac{1}{2}}\times$$
$$\times\Big\{\sum_{n\geqslant 1} 2^{-n\varepsilon} \sum_{1\leqslant k\leqslant 2^n} |\mu(\Delta^{(j)}_{kn})|^2\Big\}^{\frac{1}{2}} \leqslant$$
$$\leqslant CJ_{21}(t,x) + CJ_{22}(t,x).$$

Here

$$J_{21}(t,x) =$$
$$= \Big(\sum_{j\in\mathbb{Z}} (1+|j|)^{-2\tau}\|h(t,x,\cdot)\|^2_{\mathsf{L}_2([j,j+1])}\Big)^{\frac{1}{2}}\times$$
$$\times\Big(\sum_{j\in\mathbb{Z}} (1+|j|)^{2\tau}\sum_{n\geqslant 1} 2^{-n\varepsilon} \sum_{1\leqslant k\leqslant 2^n} |\mu(\Delta^{(j)}_{kn})|^2\Big)^{\frac{1}{2}} =$$
$$= D_1\cdot D_2,$$

and

$$J_{22}(t,x) = \Big(\sum_{j\in\mathbb{Z}} (1+|j|)^{-2\tau}\times$$
$$\times\Big(\int_0^1 \big(w_{2,[j,j+1]}(h,r)\big)^2 r^{-2\alpha-1}\,dr\Big)^{\frac{1}{2}}\Big]\Big)^{\frac{1}{2}}\times$$
$$\times\Big(\sum_{j\in\mathbb{Z}} (1+|j|)^{2\tau}\sum_{n\geqslant 1} 2^{-n\varepsilon} \sum_{1\leqslant k\leqslant 2^n} |\mu(\Delta^{(j)}_{kn})|^2\Big)^{\frac{1}{2}} =$$
$$= D_3\cdot D_2.$$

Firstly, estimate the L_2-norm of function $h(t,x,\cdot)$ for fixed parameters (t,x). We have from (10), for any $j\in\mathbb{R}$,

$$\|h(t,x,\cdot)\|_{\mathsf{L}_2([j,j+1])} = \Big(\int_j^{j+1} |h(t,x,y)|^2 dy\Big)^{\frac{1}{2}} \leqslant$$
$$\leqslant C\sqrt{t},$$

hence,

$$\|h(t,x,\cdot)\|_{\mathsf{L}_2([j,j+1])} \to 0, |x| \to \infty.$$

We obtain that series D_1 converges and tends to 0 as $|x|$ tends to ∞.

On the other hand, by [11, Lemma 3.1] series D_2 is bounded almost surely. Here we use Lemma 3.1 for functions

$$g_j(y) = 2^{-\frac{n\varepsilon}{2}}(1+|j|)^\tau \mathbb{1}_{\Delta_{kn}^{(j)}}(y), j \in \mathbb{Z}, n \in \mathbb{N}, k = \overline{1,n}.$$

We can put $u(t,x) = 0$ on the set $\{D_2 = +\infty\}$.

Thus,

$$J_{21}(t,x) \to 0, \quad |x| \to \infty. \tag{12}$$

Further we estimate function $w_{2,[j,j+1]}(h,r)$, for $r \in [0,1]$, and term $J_{22}(t,x)$. Let $v \in [0,r]$, we have

$$|h(t,x,y+v) - h(t,x,y)| =$$
$$= \Big| \int_0^t p(t,x-y-v;s)\sigma(s,y+v)\,ds -$$
$$- \int_0^t p(t,x-y;s)\sigma(s,y)\,ds \Big| \leqslant$$
$$\leqslant \int_0^t \big|p(t,x-y-v;s)\big|\big|\sigma(s,y+v)-\sigma(s,y)\big|ds +$$
$$+ \Big| \int_0^t \big(p(t,x-y-v;s) - p(t,x-y;s)\big)\sigma(s,y)ds \Big| =$$
$$= \widetilde{J}_1(t,x) + \widetilde{J}_2(t,x).$$

Next we use Assumption A6 and estimate (6) to conclude

$$\widetilde{J}_1(t,x) \leqslant L_\sigma v^{\beta(\sigma)} \int_0^t M(t-s)^{-1/2} e^{-\frac{\lambda|x-y-v|^2}{t-s}} ds \leqslant$$
$$\leqslant Cv^{\beta(\sigma)}.$$

Moreover, similarly to (11) we obtain

$$\widetilde{J}_1(t,x) \to 0, \ |x| \to \infty. \tag{13}$$

Also, by A5 and (7),

$$\widetilde{J}_2(t,x) =$$
$$= \Big| \int_0^t \int_y^{y+v} \frac{\partial p(t,x-w;s)}{\partial w} \sigma(s,y) dw ds \Big| \leqslant$$
$$\leqslant C_\sigma \int_0^t \int_y^{y+v} M(t-s)^{-1} e^{-\frac{\lambda|x-w|^2}{t-s}} dw ds.$$

According to upper bound (11) from [4] by A5 we get

$$\widetilde{J}_2(t,x) \leqslant Cv^\gamma,$$

for any $\gamma \in (0,1)$, where constant C depends on γ, M, λ and T.

Taking into account the representation

$$\widetilde{J}_2(t,x) \leqslant \int_0^t \big|p(t,x-y-v;s)\sigma(s,y)\big|ds +$$
$$+ \int_0^t \big|p(t,x-y;s)\sigma(s,y)\big|ds \leqslant C,$$

we can similary to (11) conclude that

$$\widetilde{J}_2(t,x) \to 0, \ |x| \to \infty. \tag{14}$$

Recall that $v, \gamma \in (0,1)$ and $\beta(\sigma) > 1/2$. Thus,

$$|h(t,x,y+v) - h(t,x,y)| \leqslant Cv^{\alpha_1},$$

where $\alpha_1 = \min\{\beta(\sigma), \gamma\}$. Since $\gamma \in (0,1)$ is arbitrary, the last estimate is correct for any $\alpha_1 \leqslant \beta(\sigma)$.

Therefore,

$$w_{2,[j,j+1]}(h,r) =$$
$$= \sup_{0 \leqslant v \leqslant r} \Big(\int_j^{j+1-v} |h(t,x,y+v) - h(t,x,y)|^2 \Big)^{\frac{1}{2}} \leqslant$$
$$\leqslant \sup_{0 \leqslant v \leqslant r} \Big(C|1-v|v^{2\alpha_1} \Big)^{\frac{1}{2}} \leqslant Cr^{\alpha_1}.$$

Hence there exists $\alpha \in (1/2, 1)$ such that $1/2 < \alpha < \alpha_1 \leqslant \beta(\sigma)$ and

$$\int_0^1 \big(w_{2,[j,j+1]}(h,r)\big)^2 r^{-2\alpha-1}\,dr \leqslant$$
$$\leqslant \int_0^1 Cr^{-2\alpha-1} r^{2\alpha_1} dr = C \frac{r^{2(\alpha_1-\alpha)}}{2(\alpha_1-\alpha)}\Big|_0^1 \leqslant C,$$

where C depends on $\beta(\sigma)$.

On the other hand, due to convergences (13) and (14), by the Lebesgue's dominated convergence theorem we conclude

$$\int_0^1 \big(w_{2,[j,j+1]}(h,r)\big)^2 r^{-2\alpha-1}\,dr \to 0, \ |x| \to \infty.$$

Consequently,

$$E_1 \to 0, \ |x| \to \infty,$$

and similary to (12) we get

$$J_{22}(t,x) \to 0, \quad |x| \to \infty,$$

and the same result is fulfilled for $J_2(t,x) = J_{21}(t,x) + J_{22}(t,x)$. That is why we have

$$I_3(t,x) \to 0, \quad |x| \to \infty.$$

Consider another terms of (8). For an arbitrary $y_0 \in \mathbb{R}_+$, we have

$$I_2(t,x) \leqslant$$
$$\leqslant \left| \int_0^t ds \int_{|y| \leqslant y_0} p(t, x-y; s) f(s, y, u(s,y))\, dy \right| +$$
$$+ \left| \int_0^t ds \int_{|y| > y_0} p(t, x-y; s) f(s, y, u(s,y))\, dy \right| =$$
$$= I_{21}(t,x) + I_{22}(t,x).$$

By Assumption A9 for any $\varepsilon > 0$ there exists $y_0 \in \mathbb{R}_+$, such that

$$\sup_{s \in [0,T], z \in \mathbb{R}} |f(s,y,z)| < \varepsilon, \ \forall\, |y| > y_0.$$

Let fix this y_0. Since

$$\int_{\mathbb{R}} |p(t, x-y; s)| dy \leqslant C,$$

where C depends on M, λ from (6), and on T (see [4, estimate (7)]), we can conclude that

$$I_{22}(t,x) \leqslant \varepsilon t \leqslant \varepsilon T.$$

Further by the Lebesgue's dominated convergence theorem we get

$$I_{21}(t,x) \to 0, \ |x| \to \infty.$$

Here we use the estimates

$$|p(t, x-y; s) f(s, y, u(s,y))| \leqslant$$
$$\leqslant C_f M (t-s)^{-\frac{1}{2}} e^{-\frac{\lambda |x-y|^2}{t-s}}$$

(obtained by inequality (6) and Assumption A3) and

$$e^{-\frac{\lambda |x-y|^2}{t-s}} \leqslant e^{-\frac{\lambda |y_0-y|^2}{t-s}}, \ |x| \geqslant y_0,$$

and also the arguments similar to those applied in the inequality (11).

Thus we obtain that $I_2(t,x)$ tends to zero as $|x| \to \infty$.

The same arguments and condition $A10$ for function u_0 give us convergence

$$I_1(t,x) \to 0, \ |x| \to \infty.$$

As the result, we get the statement of our Theorem.

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