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Теорія дуальності для увігнення функцій корисності в неповній моделі ринку

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O.O. Bahchedjioglou¹, post-graduate student Duality theory for concavification of utility functions in incomplete market model

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Основна мета цієї роботи полягає у доведенні існування оптимальної стратегії для стандартної та робастної задач максимізації функції корисності за умов неповної моделі ринку, коли інвестор не впевнений у ймовірнісній моделі та не сприймає як ризик, так і невизначеність моделі. Існуючі результати досліджень для строго увігнутих функцій корисності узагальнено для не увігнутих функцій корисності. Крім того, сформульовано умови, за яких оптимальна стратегія для задач з увігнутою функцією корисності збігається з оптимальною стратегією для початкової функції корисності.

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Ключові слова: функціонал максимізації функції корисності, оптимальні стратегії, оптимальні інвестиції, увігнення.

The main goal for this paper is to prove the existence of the optimal investment strategies for the standard and robust problems of maximization for the concavified utility function in an incomplete market model. We extend the existing results for strictly concave utility functions to concavification of non-concave utility functions. Moreover, we present an assumption under which the optimal strategies for concavified problems are also optimal strategies for non-concave problems.

Keywords: standard utility maximization, robust utility maximization, optimal investments, non-concave utility, concavification.

1 Introduction

The optimal investment problem is one of the most actual problem nowadays. There is a lot of aspects that can be considered in this problem such as the concavity of utility function, market model, the assumptions on the value process, the set of probability measures and the possibility of constructing the optimal investment strategies etc. The main interest in constructing the optimal investment in the very general setup, with an incomplete market, general sets of prior models and non-concave utility functions.

2 Model setup

We will consider the model as in [3] with the additional assumption that the discounted price process is locally bounded. We impose the no-arbitrage type condition that the set of all equivalent local martingale measures $\mathcal{M}_e \neq \emptyset$, but we

do not assume that the market model is complete.

We consider the discounted price process with d assets which modeled by a stochastic process $S = (S_t)_{0 \leq t \leq T}$. We assume that S is a d -dimensional locally bounded semimartingale on $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to a filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$. The pair (x, ξ) is a self-financing trading strategy, where $x \in \mathbb{R}$ is the initial wealth and $\xi = (\xi_t)_{0 \leq t \leq T}$ is a d -dimensional predictable and S is an integrable process. The corresponding value process X satisfies

$$X_t = X_0 + \int_0^t \xi_r dS_r, \quad 0 \leq t \leq T.$$

By $\mathcal{X}(x)$, $x > 0$ we denote the set of all such processes X , with $X_0 \leq x$ which are also admissible in the sense that $X_t \geq 0$, for $0 \leq t \leq T$.

Additionally, the authors of [3] impose the assumptions on the set of probability measures $\mathcal{Q}$ on $(\Omega, \mathcal{F})$.

Assumption 1. (i) $\mathcal{Q}$ is convex;

(ii) $\mathbb{P}[A] = 0$ if and only if $Q[A] = 0$ for all $Q \in \mathcal{Q}$;

(iii) The set $\mathcal{Z} := \{dQ/dP | Q \in \mathcal{Q}\}$ is closed in $L^0(\mathbb{P})$.

We add the following extra assumption to this set:

(iv) The set $\mathcal{Z}_e := \{dQ/dP | Q \in \mathcal{Q}_e\}$ is closed in $L^0(\mathbb{P})$,

where $\mathcal{Q}_e$ denotes the set of measures in $\mathcal{Q}$ that are equivalent to $\mathbb{P}$.

Let an investor have utility function $U(x)$; the conditions on $U(x)$ will be specified later. Consider the following value function of the robust utility problem:

$$u(x) := \sup_{X \in \mathcal{X}(x)} \inf_{Q \in \mathcal{Q}} E_Q[U(X_T)].$$

Denote

$$u_Q(x) := \sup_{X \in \mathcal{X}(x)} E_Q[U(X_T)],$$

the value function of the optimal investment problem.

As in [4, 5], we will consider the following “abstract version” of $\mathcal{X}(x)$:

$$C(x) := \{g \in L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}) | \\ 0 \leq g \leq X_T \text{ for some } X \in \mathcal{X}(x)\}.$$

It is easy to see that

$$\begin{aligned} u(x) &= \sup_{X \in \mathcal{X}(x)} \inf_{Q \in \mathcal{Q}} E_Q[U(X_T)] \\ &= \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}} E_Q[U(g)]; \\ u_Q(x) &= \sup_{X \in \mathcal{X}(x)} E_Q[U(X_T)] = \sup_{g \in C(x)} E_Q[U(g)]. \end{aligned}$$

3 Main result

The aim of this article is to present the optimal investment strategies for the standard as well as the robust problem of maximization the concavified utility function, i.e. $\sup_{X \in \mathcal{X}(x)} \mathbb{E}[U_c(X_T)]$ and

$\sup_{X \in \mathcal{X}(x)} \inf_{Q \in \mathcal{Q}} E_Q[U(X_T)]$. The results and proofs of the present article borrow their main ideas and techniques from [3]. The main difference is that,

contrary to [3], we do not assume the concavified utility function is strictly concave and we obtain results for a fixed probability measure. Moreover, we present a sufficient assumption for optimal strategies for concavified problems to be optimal strategies for non-concave problems as well.

Denote

$$u_P^c(x) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U_c(X_T)].$$

To avoid trivialities, throughout the article we assume that

$$u_P^c(x) < \infty \text{ for some } x > 0. \quad (1)$$

3.1 Duality theory

In this section we will work with the convex conjugate V of U :

$$V(y) = \sup_{x > 0} (U(x) - xy), \quad y > 0.$$

We do not assume that U is concave or satisfies the Inada condition in zero. As a result, in general, the concave envelope U_c is not strictly concave and the Inada condition in zero does not hold as well. Hence, the convex conjugate V in general is not smooth and not strictly decreasing. Nevertheless it follows from the [1, Lemma 2.9] that the utility U and its concave envelope U_c have the same convex conjugate and that the function V is convex, non-increasing and finite on $(0, \infty)$. Moreover, it holds that

$$\begin{aligned} U_c(x) - xy = V(y) &\iff x \in -\partial V(y) \\ &\iff y \in \partial U_c(x). \end{aligned} \quad (2)$$

The dual value function v_P for u_P is given by

$$v_P(y) := \inf_{Y \in \mathcal{Y}(y)} \mathbb{E}[V(Y_T)], \quad y > 0,$$

where the space $\mathcal{Y}(y)$ is defined as

$$\mathcal{Y}(y) = \{Y \geq 0 | Y_0 = y \text{ and } XY \text{ is a } \mathbb{P} - \text{supermartingale for all } X \in \mathcal{X}(1)\}.$$

In addition to $C(x)$, consider also the “abstract version” of $\mathcal{Y}(y)$:

$$\begin{aligned} D(y) &= \{h \in L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}) | \\ 0 \leq h \leq Y_T \text{ for some } Y \in \mathcal{Y}(y)\}. \end{aligned}$$

It is easy to see that

$$\begin{aligned} u_P(x) &= \sup_{g \in C(x)} \mathbb{E}[U(g)] \\ u_P^c(x) &= \sup_{g \in C(x)} \mathbb{E}[U_c(g)] \\ v_P(y) &= \inf_{h \in D(y)} \mathbb{E}[V(h)]. \end{aligned}$$

Moreover, the optimal g or h (if it exists) corresponds to some $X \in \mathcal{X}(x)$ or $Y \in \mathcal{Y}(y)$.

Recall that a subset S of $L_+^0(\Omega, \mathcal{F}_T, \mathbb{P})$ is called *solid* if from $0 \leq f \leq g$ and $g \in S$ it follows that $f \in S$.

By [4, Proposition 3.1], we have

- (i) $C(x)$ and $D(y)$ are subsets of $L_+^0(\Omega, \mathcal{F}_T, \mathbb{P})$ which are convex, solid and closed in the topology of convergence in measure.

(ii)

$$\begin{aligned} g \in C(x) &\iff g \geq 0 \text{ and } \sup_{h \in D(y)} \mathbb{E}[hg] \leq xy \\ h \in D(y) &\iff h \geq 0 \text{ and } \sup_{g \in C(x)} \mathbb{E}[hg] \leq xy. \end{aligned} \quad (3)$$

- (iii) $C(x)$ is a bounded subset of $L_+^0(\Omega, \mathcal{F}_T, \mathbb{P})$, containing the constant function x .

3.2 Optimal investments for the standard concavified utility functional problem

In this section we are going to present the optimal investment strategies for the problem of maximization the standard concavified utility function, i.e. $\sup_{X \in \mathcal{X}(x)} \mathbb{E}[U_c(X_T)]$.

Assume that

$$U_c(0) \geq 0, \quad (4)$$

Then, for all $y > 0$,

$$\begin{aligned} V(y) &= \sup_{x > 0} (U_c(x) - xy) \geq U_c(0) \geq 0 \\ V_n(y) &= \sup_{0 < x \leq n} (U_c(x) - xy) \geq U_c(0) \geq 0. \end{aligned}$$

From the last inequality and the assumption (1), appealing to the proof of [4, Lemma 3.4], we obtain that

$$v_P(y) = \sup_{x > 0} (u_P^c(x) - xy) \text{ for each } y > 0. \quad (5)$$

Theorem 2. Let (1) and (4) hold.

Then, for any $x_0 > 0$, there exists some $\hat{g} \in C(x_0)$, $\hat{y} > 0$ and $\hat{h} \in D(\hat{y})$ such that

- (i) $u_P^c(x_0) = v_P(\hat{y}) + x_0\hat{y}$,
(ii) $u_P^c(x_0) = \inf_{y > 0} (v_P(y) + x_0y)$.

If, in addition, $v_P(y) < \infty$ for all $y > 0$, then

- (iii) $v_P(\hat{y}) = \mathbb{E}[V(\hat{h})]$,
(iv) $u_P^c(x_0) = \mathbb{E}[U_c(\hat{g})]$.

Moreover, $v_P(\cdot)$ is convex.

Proof. Fix $x_0 > 0$. For all $y > 0$, we have

$$u_P^c(x_0) \leq \sup_{x > 0} [u_P^c(x) - xy] + x_0y. \quad (6)$$

Let us prove that there exists $\hat{y} > 0$, such that for all $x > 0$, the following holds:

$$u_P^c(x_0) \geq u_P^c(x) - x\hat{y} + x_0\hat{y}, \quad (7)$$

equivalently,

$$u_P^c(x) - u_P^c(x_0) \leq \hat{y}(x - x_0).$$

Since u_P^c is concave (see [3, Lemma 3.1 (b)]), such $\hat{y}$ exists. Moreover, from the inequality $u_P^c(x_0^+) \leq u_P^c(x_0^-)$ it follows that $\hat{y} \in [u_P^c(x_0^+), u_P^c(x_0^-)]$.

Then, using (7),

$$u_P^c(x_0) \geq \sup_{x > 0} [u_P^c(x) - x\hat{y}] + x_0\hat{y}.$$

Therefore, noting (6), we have

$$u_P^c(x_0) = \sup_{x > 0} [u_P^c(x) - x\hat{y}] + x_0\hat{y}.$$

As a result, (5) implies (i).

To prove (ii), observe that from (5) we have for all $x_0, y > 0$

$$u_P^c(x_0) \leq v_P(y) + x_0y.$$

Hence,

$$u_P^c(x_0) \leq \inf_{y > 0} (v_P(y) + x_0y).$$

Using (i), we get

$$u_P^c(x_0) \geq \inf_{y > 0} (v_P(y) + x_0y),$$

which yields (ii).

From the definition of $v_P(\hat{y})$ it follows that there exists a sequence $(h_n) \in D(\hat{y})$ such that

$$\lim_{n \rightarrow \infty} \mathbb{E}[V(h_n)] = \inf_{h \in D(\hat{y})} \mathbb{E}[V(h)].$$

Due to the Kolmos-type argument [2, Lemma A1.1], there is sequence $(\hat{h}_n)_{n \in \mathbb{N}} \in \text{conv}(h_n, h_{n+1}, \dots)$ such that $\hat{h}_n$ converges a.s. to some $\hat{h} \geq 0$.

Thanks to the convexity and closedness with respect to convergence in measure of $D(\hat{y})$, $D(\hat{y})$ is closed under countable convex combinations, hence $(\hat{h}_n) \in D(\hat{y})$. Moreover, since $D(\hat{y})$ is closed in the topology of convergence in measure, we have that $\hat{h} \in D(\hat{y})$.

Further, for each $n \in \mathbb{N}$, there exists a sequence $\alpha_i^n \in \mathbb{R}^+$ such that $\sum_{i \in \mathbb{N}} \alpha_i^n = 1$ and

$$\hat{h}_n = \sum_{i \in \mathbb{N}} \alpha_i^n h_{n+i-1}.$$

Using the continuity of V on $(0, \infty)$ (which follows from its convexity) and the assumptions of the theorem, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mathbb{E}[V(\hat{h}_n)] &= \limsup_{n \rightarrow \infty} \mathbb{E} \left[V \left(\sum_{i \in \mathbb{N}} \alpha_i^n h_{n+i-1} \right) \right] \\ &\leq \limsup_{n \rightarrow \infty} \sum_{i \in \mathbb{N}} \alpha_i^n \cdot \mathbb{E}[V(h_{n+i-1})] = \inf_{h \in D(\hat{y})} \mathbb{E}[V(h)]. \end{aligned}$$

Since $\liminf_{n \rightarrow \infty} \mathbb{E}[V(\hat{h}_n)] \leq \limsup_{n \rightarrow \infty} \mathbb{E}[V(\hat{h}_n)]$ and $\hat{h}_n \in D(\hat{y})$, we obtain

$$\lim_{n \rightarrow \infty} \mathbb{E}[V(\hat{h}_n)] = \inf_{h \in D(\hat{y})} \mathbb{E}[V(h)].$$

Noting that $V(y) \geq 0$, $y > 0$, we have

$$\begin{aligned} \mathbb{E}[V(\hat{h})] &= \mathbb{E}[\liminf_{n \rightarrow \infty} V(\hat{h}_n)] \\ &\leq \liminf_{n \rightarrow \infty} \mathbb{E}[V(\hat{h}_n)] = \inf_{h \in D(\hat{y})} \mathbb{E}[V(h)], \end{aligned}$$

arriving at (iii).

Since $v_P(y) < \infty$ for $y > 0$, it follows from (ii) that

$$\lim_{x \rightarrow \infty} \frac{u_P^c(x)}{x} = 0.$$

Hence, (iv) is an implication of [5, Lemma 1]. It remains to show that $v_P(\cdot)$ is convex. For $0 < \lambda < 1$ consider

$$\begin{aligned} \lambda v_P(y_1) + (1 - \lambda) v_P(y_2) &= \lambda \mathbb{E}[V(\hat{h}_1)] + (1 - \lambda) \mathbb{E}[V(\hat{h}_2)] \\ &\geq \mathbb{E}[V(\lambda \hat{h}_1 + (1 - \lambda) \hat{h}_2)] \\ &\geq \inf_{h \in D(\lambda y_1 + (1 - \lambda) y_2)} \mathbb{E}[V(h)] = v_P(\lambda y_1 + (1 - \lambda) y_2); \end{aligned}$$

here we have used that $(\lambda \hat{h}_1 + (1 - \lambda) \hat{h}_2) \in D(\lambda y_1 + (1 - \lambda) y_2)$. for $\hat{h}_1 \in D(y_1)$ and $\hat{h}_2 \in D(y_2)$. The proof is now complete

Theorem 3. Assume that $v_P(y) < \infty$ for all $y > 0$ and that (1), (4) hold.

Then, for any $x > 0$, there exists an optimal strategy $\hat{X} \in \mathcal{X}(x)$ such that

$$u_P^c(x) = \mathbb{E}[U_c(\hat{X}_T)].$$

There also exists some $\hat{y} > 0$ and $\hat{Y} \in \mathcal{Y}(\hat{y})$ such that

$$v_P(\hat{y}) = \mathbb{E}[V(\hat{Y}_T)] \text{ and } u_P^c(x) = v_P(\hat{y}) + x\hat{y}.$$

Moreover, $\hat{X}_T \in -\partial V(\hat{Y}_T)$ and $\hat{X}\hat{Y}$ is a martingale.

Proof. By Theorem 2 (iv), for all $x > 0$, there exists an optimal strategy $\hat{X} \in \mathcal{X}(x)$ (with $\hat{g} = \hat{X}_T$). Also from Theorem 2 (i), (iii) there exists some $\hat{y} > 0$ and $\hat{Y} \in \mathcal{Y}(\hat{y})$ (with $\hat{h} = \hat{Y}_T$) such that

$$\begin{aligned} u_P^c(x) &= \mathbb{E}[U_c(\hat{X}_T)] \\ v_P(\hat{y}) &= \mathbb{E}[V(\hat{Y}_T)] \\ u_P^c(x) &= v_P(\hat{y}) + x\hat{y}. \end{aligned}$$

By the definition of the convex conjugate V ,

$$V(\hat{h}) + \hat{g}\hat{h} - U_c(\hat{g}) \geq 0.$$

Hence,

$$\begin{aligned} 0 &\leq \mathbb{E}[V(\hat{h}) + \hat{g}\hat{h} - U_c(\hat{g})] = v_P(\hat{y}) + \mathbb{E}[\hat{g}\hat{h}] - u_P^c(x) \\ &\stackrel{(3)}{\leq} v_P(\hat{y}) + x\hat{y} - u_P^c(x) = 0. \end{aligned} \quad (8)$$

Thus,

$$V(\hat{h}) + \hat{g}\hat{h} - U_c(\hat{g}) = 0.$$

From (2) we have

$$\hat{g} \in -\partial V(\hat{h}).$$

Equivalently,

$$\hat{X}_T \in -\partial V(\hat{Y}_T).$$

But (8) implies that $\mathbb{E}[\hat{X}_T \hat{Y}_T] = \mathbb{E}[\hat{g}\hat{h}] = x\hat{y}$.

As a result, since $\hat{X}\hat{Y}$ is a $\mathbb{P}$ -supermartingale for all $X \in \mathcal{X}(x)$, we have

$$\mathbb{E}[\hat{X}_T \hat{Y}_T | \mathcal{F}_s] \leq \hat{X}_s \hat{Y}_s, s \leq T.$$

This leads to

$$\mathbb{E}[\widehat{X}_T \widehat{Y}_T] \leq \mathbb{E}[\widehat{X}_s \widehat{Y}_s] \leq \widehat{X}_0 \widehat{Y}_0 = x \widehat{y}.$$

Consequently, all the above inequalities turn to equalities, and for all $0 \leq s \leq T$ it holds that

$$\mathbb{E}[\widehat{X}_s \widehat{Y}_s] = \widehat{X}_0 \widehat{Y}_0 = x \widehat{y}. \quad (9)$$

Assume that there exists $A \in \mathcal{F}$ such that $\mathbb{P}(A) > 0$ and $\mathbb{E}[\widehat{X}_t \widehat{Y}_t | \mathcal{F}_s] < \widehat{X}_s \widehat{Y}_s$ on A for some $0 \leq s \leq t$. Then we get $\mathbb{E}[\widehat{X}_t \widehat{Y}_t] < \mathbb{E}[\widehat{X}_s \widehat{Y}_s]$, which contradicts (9). Therefore, $\mathbb{E}[\widehat{X}_t \widehat{Y}_t | \mathcal{F}_s] = \widehat{X}_s \widehat{Y}_s$ $\mathbb{P}$ -a.s., for all $0 \leq s \leq t$.

3.3 Optimal investments for the robust concavified utility functional problem

In this section we are going to present the optimal investment strategies for the problem of maximization the robust concavified utility function, i.e.

$$\sup_{X \in \mathcal{X}(x)} \inf_{Q \in \mathcal{Q}} E_Q[U_c(X_T)].$$

For a strictly increasing, strictly concave, continuously differentiable utility function, defined on the domain $(0, \infty)$ and satisfying the Inada conditions, this problem was solved by authors of [3] in general incomplete market. The concavified utility function U_c does not satisfy these assumptions. However, U_c is strictly increasing, concave, continuously differentiable on the $(0, \infty)$ and satisfies the Inada conditions in the ∞ .

Moreover, we will assume that (4) holds, and, hence, $V(y) \geq 0$ and $V_n(y) = \sup_{0 < x \leq n} (U_c(x) - xy) \geq 0$ for all $y > 0$.

In this section we will often use the results of the [3], so we present the notation that they used.

Consider first the sets

$$\begin{aligned} \mathcal{Y}_Q(y) &= \{Y \geq 0 | Y_0 = y \text{ and } XY \text{ is a} \\ &\quad Q - \text{supermartingale for all } X \in \mathcal{X}(1)\} \\ D_Q(y) &= \{h \in L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}) | 0 \leq h \leq Y_T \\ &\quad \text{for some } Y \in \mathcal{Y}_Q(y)\} \end{aligned}$$

Noting [3, Lemma 3.4]

$$u^c(x) = \inf_{Q \in \mathcal{Q}_e} u_Q^c(x). \quad (10)$$

Denote the dual value function v_Q by

$$v_Q(y) := \inf_{Y \in \mathcal{Y}_Q(y)} E_Q[V(Y_T)].$$

It is easy to see that $v_Q(y) = \inf_{h \in D_Q(y)} E_Q[V(h)]$.

The dual value function of the robust problem defined by

$$v(y) := \inf_{Q \in \mathcal{Q}_e} v_Q(y).$$

Assume that for all $Q \in \mathcal{Q}_e$,

$$u_Q^c(x) < \infty \text{ for some } x > 0. \quad (11)$$

Theorem 4. Let $Q \in \mathcal{Q}_e$ and assume that (11) and (4) hold.

Then, for any $x_0 > 0$, there exist some $\widehat{g} \in C(x_0)$, $\widehat{y} > 0$ and $\widehat{h} \in D_Q(\widehat{y})$ such that

$$(i) \quad u_Q^c(x_0) = v_Q(\widehat{y}) + x_0 \widehat{y},$$

$$(ii) \quad u_Q^c(x_0) = \inf_{y > 0} (v_Q(y) + x_0 y)$$

If in addition $v_Q(y) < \infty$ for all $y > 0$, then

$$(iii) \quad v_Q(\widehat{y}) = E_Q[V(\widehat{h})],$$

$$(iv) \quad u_Q^c(x_0) = E_Q[U_c(\widehat{g})].$$

Moreover, $v_Q(\cdot)$ is convex.

Proof. The proof is similar to the proof of Theorem 2.

Now we are going to present a similar theorem, but for the robust problem. To ensure that the dual value function is finite, we introduce the following assumption:

$$v_Q(y) < \infty \text{ for all } Q \in \mathcal{Q}_e \text{ and all } y > 0. \quad (12)$$

Theorem 5. Suppose that Assumption 3.1, (11) (4) and (12) hold. Additionally, assume that $\lim_{x \rightarrow \infty} \frac{u_Q^c(x)}{x} = 0$, for each $Q \in \mathcal{Q}_e$.

Then, for any $x_0 > 0$, there exist some $\widehat{Q} \in \mathcal{Q}_e$, $\widehat{y} > 0$ and $\widehat{h} \in D_{\widehat{Q}}(\widehat{y})$ such that

$$(i) \quad u^c(x_0) = u_{\widehat{Q}}^c(x_0),$$

$$(ii) \quad u^c(x_0) = \inf_{y > 0} (v(y) + x_0 y),$$

$$(iii) \quad u^c(x_0) = v(\widehat{y}) + x_0 \widehat{y},$$

$$(iv) \quad v(\widehat{y}) = v_{\widehat{Q}}(\widehat{y}) = E_{\widehat{Q}}[V(\widehat{h})].$$

Proof. The proof of (i) follows from [3, Lemma 3.4].

Note that $\widehat{Z}$, which is the density corresponding to $\widehat{Q}$, is an almost sure limit of a sequence $Z_n \in \mathcal{Z}_e$ and also that $u_{Q_n}^c(x_0) \rightarrow u^c(x_0)$ (see [3, Lemma 4.1 (a)]). Moreover, it follows from the first part of the proof of [3, Lemma 4.1 (a)] that

$$u^c(x_0^+) \leq \liminf_{n \rightarrow \infty} u_{Q_n}^c(x_0^+) \leq \limsup_{n \rightarrow \infty} u_{Q_n}^c(x_0^-) \leq u^c(x_0^-). \quad (13)$$

From Theorem 4 (ii) we deduce that for all $Q \in \mathcal{Q}_e$ and $y > 0$,

$$u_Q^c(x_0) \leq v_Q(y) + x_0 y.$$

Consequently,

$$\inf_{Q \in \mathcal{Q}_e} u_Q^c(x_0) \leq \inf_{Q \in \mathcal{Q}_e} (v_Q(y) + x_0 y).$$

Hence, it follows from the (10) and definition of the dual value function for robust problem that

$$u^c(x_0) \leq v(y) + x_0 y.$$

Since $y > 0$ is arbitrary, we have

$$u^c(x_0) \leq \inf_{y > 0} (v(y) + x_0 y). \quad (14)$$

To prove the opposite inequality, consider sequence Q_n (or, equivalently, Z_n) from the first part of the proof, and write

$$u^c(x_0) = \lim_{n \rightarrow \infty} u_{Q_n}^c(x_0).$$

From the Theorem 4 (i) it follows that for each Q_n there exists $\widehat{y}_n \in [u_{Q_n}^c(x_0^+), u_{Q_n}^c(x_0^-)]$ such that

$$u_{Q_n}^c(x_0) = v_{Q_n}(\widehat{y}_n) + x_0 \widehat{y}_n.$$

Passing, if necessary, to a subsequence, we can assume that sequence $(\widehat{y}_n)$ converges to some $\widehat{y} \in [u^c(x_0^+), u^c(x_0^-)]$, due to (13). Since u^c is strictly increasing, we have $\widehat{y} > 0$. Therefore,

$$\begin{aligned} u^c(x_0) &= \lim_{n \rightarrow \infty} u_{Q_n}^c(x_0) = \lim_{n \rightarrow \infty} (v_{Q_n}(\widehat{y}_n) + x_0 \widehat{y}_n) \\ &\geq \limsup_{n \rightarrow \infty} (v(\widehat{y}_n) + x_0 \widehat{y}_n) \geq \inf_{y > 0} (v(y) + x_0 y). \end{aligned} \quad (15)$$

In view of (14), we obtain (ii).

In order to prove (iii), we will use the results of the [3]. Thanks to (4), $V(y) \geq 0$ and $V_n(y) \geq 0$. Due to assumption (12), we have $v(y) < \infty$. Hence, it follows from the [3, Lemma 3.7] that $v(y)$ is convex, thus it is continuous on $(0, \infty)$. Since $\widehat{y}_n$

converges almost surely to $\widehat{y} > 0$, we derive from (15) that

$$u^c(x_0) = \lim_{n \rightarrow \infty} (v(\widehat{y}_n) + x_0 \widehat{y}_n) = v(\widehat{y}) + x_0 \widehat{y}. \quad (16)$$

Thus, it remains to show (iv). Combining (15) and (16), we get

$$\lim_{n \rightarrow \infty} v_{Q_n}(\widehat{y}_n) = v(\widehat{y}).$$

Theorem 4 (iii) yields for each $n \in \mathbb{N}$ the existence of $h'_n \in D_{Q_n}(\widehat{y}_n)$ such that

$$v_{Q_n}(\widehat{y}_n) = E_{Q_n}[V(h'_n)].$$

This is equivalent to the existence of $h_n \in D(\widehat{y}_n)$ such that

$$v_{Q_n}(\widehat{y}_n) = \mathbb{E} \left[Z_n V \left(\frac{h_n}{Z_n} \right) \right],$$

where $Z_n \in \mathcal{Z}_e$ are the densities which correspond to the measures Q_n .

Due to the Kolmos-type argument, see [2, Lemma A1.1], there exists a sequence $(\widetilde{h}_n)_{n \in \mathbb{N}} \in \text{conv}(h_n, \widetilde{h}_{n+1}, \dots)$ such that $\widetilde{h}_n$ converges $\mathbb{P}$ -a.s. to some $\widetilde{h}$.

Now we will show that $\widetilde{h} \in D(\widehat{y})$. For each $n \in \mathbb{N}$, there is a sequence of $\alpha_i^n \in [0, 1]$ such that $\sum_i \alpha_i^n = 1$ and $\widetilde{h}_n = \sum_i \alpha_i^n h_{n+i-1}$. Recalling (3), we obtain that

$$h_n \geq 0 \text{ and } \sup_{g \in C(x)} \mathbb{E}[h_n g] \leq x \widehat{y}_n. \quad (17)$$

Now for all $g \in C(x)$ we have

$$\begin{aligned} \mathbb{E}[\widetilde{h} g] &= \mathbb{E} \left[\lim_{n \rightarrow \infty} \sum_i \alpha_i^n h_{n+i-1} g \right] \leq \liminf_{n \rightarrow \infty} \sum_i \alpha_i^n \mathbb{E}[h_{n+i-1} g] \\ &\stackrel{(17)}{\leq} \liminf_{n \rightarrow \infty} \sum_i \alpha_i^n x \widehat{y}_{n+i-1} \\ &= x \liminf_{n \rightarrow \infty} \sum_i \alpha_i^n \widehat{y}_{n+i-1} = x \lim_{n \rightarrow \infty} \widehat{y}_n = x \widehat{y} \end{aligned}$$

Therefore, since $\widetilde{h} \geq 0$, it follows from (3) that $\widetilde{h} \in D(\widehat{y})$. Hence, it follows from the proof of [3, Lemma 3.7] that

$$v(\widehat{y}) = \mathbb{E} \left[\widehat{Z} V \left(\frac{\widetilde{h}}{\widehat{Z}} \right) \right].$$

This is equivalent to

$$v(\widehat{y}) = \mathbb{E}[\widehat{Z} V(\widehat{h})] = E_{\widehat{Q}}[V(\widehat{h})], \quad (18)$$

for $\hat{h} \in D_{\hat{Q}}(\hat{y})$.

From the definitions we have

$$v(\hat{y}) \leq v_{\hat{Q}}(\hat{y}) \leq E_{\hat{Q}}[V(\hat{h})],$$

for $\hat{Q} \in \mathcal{Q}_e, \hat{y} > 0$ and $\hat{h} \in D_{\hat{Q}}(\hat{y})$.

Thus, in view of (18), we arrive at (iv).

Theorem 6. Suppose that all assumptions of Theorem 5 hold.

Then, for any $x > 0$, there exist some $\hat{X} \in \mathcal{X}(x)$ and $\hat{Q} \in \mathcal{Q}_e$ such that

$$u^c(x) = \inf_{Q \in \mathcal{Q}} E_Q[U_c(\hat{X}_T)] = E_{\hat{Q}}[U_c(\hat{X}_T)] = u_{\hat{Q}}^c(x) \quad (19)$$

There also exist some $\hat{y} > 0$ and $\hat{Y} \in \mathcal{Y}_{\hat{Q}}(\hat{y})$ such that

$$v(\hat{y}) = E_{\hat{Q}}[V(\hat{Y}_T)] \text{ and } u^c(x) = v(\hat{y}) + x\hat{y}. \quad (20)$$

Moreover, $\hat{X}_T \in -\partial V(\hat{Y}_T)$. Also, $\hat{X}\hat{Y}$ is a $\hat{Q}$ -martingale.

Proof. The equalities (19) follow from [3, Theorem 2.6 and Lemma 4.1 (a)] for $\hat{X}_T = \hat{g}$, while (20) follows from Theorem 5, with $x_0 = x$ and $\hat{Y}_T = \hat{h}$.

To prove the remaining statements, consider, as in [3, proof of Theorem 2.6], $V(\hat{h}) + \hat{g}\hat{h} - U_c(\hat{g})$. From the definition of V , we have that

$$V(\hat{h}) + \hat{g}\hat{h} - U_c(\hat{g}) \geq 0. \quad (21)$$

Since $\hat{h} \in D_{\hat{Q}}(\hat{y})$, we have that for $\tilde{h} = \hat{h}\hat{Z} \in D(\hat{y})$ the following holds

$$E_{\hat{Q}}[\hat{g}\tilde{h}] = \mathbb{E}[\hat{g}\tilde{h}\mathbb{I}_{\{\hat{Z}>0\}}].$$

Hence,

$$\begin{aligned} E_{\hat{Q}}[V(\hat{h}) + \hat{g}\hat{h} - U_c(\hat{g})] &= v(\hat{y}) + \mathbb{E}[\hat{g}\tilde{h}\mathbb{I}_{\{\hat{Z}>0\}}] - u^c(x) \\ &\stackrel{(3)}{\leq} v(\hat{y}) + x\hat{y} - u^c(x) = 0 \end{aligned} \quad (22)$$

Thus, noting (21) we have that $V(\hat{h}) + \hat{g}\hat{h} - U_c(\hat{g}) = 0$. From (2) we get

$$\hat{g} \in -\partial V(\hat{h}),$$

equivalently,

$$\hat{X}_T \in -\partial V(\hat{Y}_T).$$

Observe that (22) implies $E_{\hat{Q}}[\hat{X}_T\hat{Y}_T] = E_{\hat{Q}}[\hat{g}\hat{h}] = x\hat{y}$. Hence, the proof of fact that $\hat{X}\hat{Y}$ is a $\hat{Q}$ -martingale is similar to the proof of Theorem 3.

3.4 Optimal investments for the problem of maximization the non-concave utility functional

The goal of this section is to present assumptions, under which the solution to the problem of maximization of the concavified utility function will be also the solution to the problem of maximization of the non-concave utility function. To this end, we will use the result of [1, Lemma 5.7].

The following theorem works for both the standard utility maximization problem as well as for the robust utility maximization problem.

Theorem 7. Suppose that Assumptions 3.1 and Assumption 5.1 hold.

Additionally, assume that the optimal solution g^* for concavified problem (either standard: $\sup_{g \in C(x)} \mathbb{E}[U_c(g)]$ or robust: $\sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}} E_Q[U_c(g)]$) is such that $g^* \in -\partial V(\lambda Z)$, where $Z \in \mathcal{Z}$ has a continuous distribution.

Then, g^* is also an optimal solution for the corresponding non-concave problem.

Proof. For the standard utility maximization problem, the proof is the same as for the [1, Lemma 5.7].

To prove the theorem for the robust utility maximization problem, note that it follows from the [1, Lemma 5.7] that under assumptions of the theorem $\mathbb{P}[g^* \in \{U < U_c\}] = 0$.

Thus,

$$\inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g^*)] = \inf_{Q \in \mathcal{Q}_e} E_Q[U(g^*)]. \quad (23)$$

Since g^* is an optimal solution for the concavified problem, we get

$$\sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}} E_Q[U_c(g)] = \inf_{Q \in \mathcal{Q}} E_Q[U_c(g^*)].$$

Noting that $\mathcal{Q}_e \subseteq \mathcal{Q}$, we have that

$$\begin{aligned} \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}} E_Q[U_c(g)] &= \inf_{Q \in \mathcal{Q}} E_Q[U_c(g^*)] \leq \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g^*)] \\ &\stackrel{(23)}{=} \inf_{Q \in \mathcal{Q}_e} E_Q[U(g^*)] \leq \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U(g)]. \end{aligned} \quad (24)$$

From [3, Proof of Lemma 3.4] and the fact that $U \leq U_c$, we obtain that

$$\begin{aligned} \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}} E_Q[U_c(g)] &\geq \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}} E_Q[U(g)] \\ &= \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U(g)]. \end{aligned}$$

Thus, all inequalities in (24) are equalities. Hence, the argument of [3, Lemma 3.4] yields

$$\sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}} E_Q[U(g)] = \inf_{Q \in \mathcal{Q}} E_Q[U(g^*)],$$

which concludes the proof.

4 Conclusions

In this paper, we proved the existence of the optimal investment strategies for standard as well as

the robust maximization problem in incomplete market model. We considered the concavified utility function, which is not necessarily strictly concave. Also we presented assumptions under which the optimal investments for concavified problems are the optimal investments for non-concave utility function, to which this concavification corresponds.

In the future, it will be really helpful to study further the optimal investments in incomplete market for non-concave utility functions.

Список використаних джерел

1. Reichlin, Christian Rochus August: "Utility Maximization with a Given Pricing Measure When the Utility Is Not Necessarily Concave"; Mathematics and Financial Economics, Volume 7, Issue 4, pp 531-556, 2013
2. Delbaen, F., Schachermayer, W. A general version of the fundamental theorem of asset pricing. Math. Ann. 300 (1994), no. 3, 463-520.
3. Schied, Alexander; Wu, Ching-Tang. Duality theory for optimal investments under model uncertainty.
4. Kramkov, D., Schachermayer, W. The asymptotic elasticity of utility functions and optimal investment in incomplete markets. Ann. Appl. Probab. 9, no. 3, 904-950 (1999).
5. Kramkov, D., Schachermayer, W. Necessary and sufficient conditions in the problem of optimal investment in incomplete markets. Ann. Appl. Probab., Vol. 13, no. 4 (2003).

References

1. REICHLIN, CHRISTIAN ROCHUS AUGUST: "Utility Maximization with a Given Pricing Measure When the Utility Is Not Necessarily Concave"; Mathematics and Financial Economics, Volume 7, Issue 4, pp 531-556, 2013
2. DELBAEN, F., SCHACHERMAYER, W. A general version of the fundamental theorem of asset pricing. Math. Ann. 300 (1994), no. 3, 463-520.
3. SCHIED, ALEXANDER; WU, CHING-TANG. Duality theory for optimal investments under model uncertainty.
4. KRAMKOV, D., SCHACHERMAYER, W. The asymptotic elasticity of utility functions and optimal investment in incomplete markets. Ann. Appl. Probab. 9, no. 3, 904-950 (1999).
5. KRAMKOV, D., SCHACHERMAYER, W. Necessary and sufficient conditions in the problem of optimal investment in incomplete markets. Ann. Appl. Probab., Vol. 13, no. 4 (2003).

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