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Група автоморфізмів варіанта решітки розбиттів скінченної множини

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Automorphism group of the variant of the lattice of partitions of a finite set

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У роботі вивчаються групи автоморфізмів варіантів напівгрупи решітки розбиттів скінченної множини. Доведено, що група автоморфізмів варіанта напівгрупи розбиттів скінченної множини ізоморфна узагальненому вінцевому добутку, в якому перший множник є прямим добутком вінцевих добутків, які задаються блоками, різних потужностей, розбиття, яке породжує варіант, а другий деякому набору симетричних груп.

Ключові слова: напівгрупа, варіант, решітка, група автоморфізмів .

In this paper we consider variants of the lattice of partitions of a finite set and study automorphism groups of this variants. We obtain irreducible generating sets for of the lattice of partitions of a finite set.

We prove that the automorphism group of the variant of the lattice of partitions of a finite set is a natural generalization of the wreath product. The first multiplier of this generalized wreath product is the direct product of the wreaths products, such that depends on the type of the variant generating partition and the second is defined by the certain set of symmetric groups.

Key Words: semigroup, variant, lattice, automorphism group.

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Introduction

For any semigroup S with the fixed element $a \in S$ and arbitrary elements $x, y \in S$ we set $x *_a y = xay$. The operation $*_a$ defined on S by this equality is associative. This operation $*_a$ is called sandwich operation and the semigroup $(S, *_a)$ is called variant of S or a sandwich semigroup.

Lyapin was one of the first researchers who establish study of variants. In the monograph [1] he studied variants of transformation semigroups. Further various authors studied other types of semigroups, for example, papers [2] [3] [4] [5], chapter 13 in monograph [6] and references in this chapter. The study of transformation semigroups continues , for example, in [9].

We study variants of lattices which are considered as semigroups with respect to the operation $\wedge$. Here we set $\wedge$ as a greatest lower bound of two elements.

Let the *partition* of the set A on k blocks be any family of nonempty disjoined sets $\{A_1, \ldots, A_k\}$ such that $A = A_1 \dot\cup \cdots \dot\cup A_k$. We call sets $A_1, \ldots, A_k$ *blocks* of the partition. The orders of blocks is not significant. The set of all partitions of the set A is denoted by by $\mathrm{Part}(A) = \bigcup_{k\geqslant 1} \mathrm{Part}_k(A)$. If $\mathsf{N} = \{1, 2, \ldots, n\}$ then instead of $\mathrm{Part}(\mathsf{N})$ we use the notation Part_n.

We call partition σ *granulation* or *subpartition* of a partition τ (denote $\sigma \leqslant \tau$) if each block of the partition τ is a union of blocks from partition σ (that is, each block of the partition σ belongs to some block of the partition τ). The relation $\leqslant$ is a partial order on the set $\mathrm{Part}(A)$ and the partially ordered set $(\mathrm{Part}(A), \leqslant)$ is a complete lattice.

Each lattice can be considered as a semigroup with respect to the operation $\wedge$. Note, that $\wedge$ if and only if $a \wedge b = a$. Furthermore, a transformation $\varphi : L \to L$ is an automorphism of a lattice L as a only if $a \wedge b = a$. Furthermore, a transformation $\varphi : L \to L$ is an automorphism of a lattice L as a partially ordered set $(L, \leqslant)$ if and only if it is an automorphism of L as a semigroup $(L, \wedge)$.

In this paper we study groups of automorphism of the semigroup $(\mathrm{Part}_n, \wedge)$ variants and further we will denote this semigroup just by Part_n.

Variants of the semigroup Part_n

Let Part_n be a lower semilattice. For any $\rho \in \mathrm{Part}_n$ by the Δ_ρ we denote the lower cone of the element ρ: $\Delta_\rho = \{\tau \in \mathrm{Part}_n | \tau \wedge \rho = \tau\}$. The lower cone Δ_ρ of the element ρ coincides with the interval $[0, \rho]$. Moreover if ρ is a partition of the set N on the blocks $M_1, \dots, M_k$ with powers $m_1, \dots, m_k$ respectively, then $[0, \rho] \simeq \mathrm{Part}_{m_1} \times \dots \times \mathrm{Part}_{m_k}$.

Lemma 1. *Each set of variant* $(\mathrm{Part}_n, *_\rho)$ *generators contains a set* $\mathrm{Part}_n \setminus [0, \rho]$.

Доведення. Let element τ be decomposable in the variant $(\mathrm{Part}_n, *_\rho)$ then there exist $\mu, \nu \in \mathrm{Part}_n$, such that $\tau = \mu *_\rho \nu = \mu \wedge \rho \wedge \nu$. It follows that

$$\tau \wedge \rho = \mu \wedge \rho \wedge \nu \wedge \rho = \mu \wedge \rho \wedge \nu = \tau, \quad (1)$$

hence $\tau \in [0, \rho]$. Thus all elements from $\mathrm{Part}_n \setminus [0, \rho]$ are non decomposable, that is why these elements belong to each generating system. □

Note that if $\rho = 1$, then the variant $(\mathrm{Part}_n, *_\rho)$ coincides with the semigroup Part_n.

Proposition 1. *If $\rho \neq 1$, then the unique irreducible generating set of the variant* $(\mathrm{Part}_n, *_\rho)$ *is the set* $\mathrm{Part}_n \setminus [0, \rho]$.

Доведення. By the Lemma 1 each generating system of the variant $(\mathrm{Part}_n, *_\rho)$ must contain $\mathrm{Part}_n \setminus [0, \rho]$. Thus it is enough to show that the set $\mathrm{Part}_n \setminus [0, \rho]$ is a generating set of the variant $(\mathrm{Part}_n, *_\rho)$. To show this it is enough to prove that each element from $[0, \rho]$ is reducible into product of elements from $\mathrm{Part}_n \setminus [0, \rho]$.

Let the partition ρ be of the next form $\mathsf{N} = M_1 \cup \dots \cup M_k$, $k \geqslant 2$. Obviously, that $\rho = 1 *_\rho 1$. Consider an arbitrary partition $\tau < \rho$. Let this partition have the form $\mathsf{N} = \left(\bigcup_{i=1}^{m_1} M_{1i}\right) \bigcup \dots \bigcup \left(\bigcup_{i=1}^{m_k} M_{ki}\right)$, where $\bigcup_{i=1}^{m_j} M_{ji} = M_j$. Without loss of generality we set $m_1 \geqslant 2$. Let partitions μ and ν be obtained from τ by "gluing"blocks M_{11} with M_{21} and blocks M_{1m_1} with M_{21} respectively. Then partitions μ and ν belong to the set $\mathrm{Part}_n \setminus [0, \rho]$ and the equality $\tau = \mu *_\rho \nu$ holds. □

Lemma 2. *An element $\tau \in \mathrm{Part}_n$ is idempotent of the semigroup* $(\mathrm{Part}_n, *_\rho)$ *if and only if* $\tau \in [0, \rho]$.

Доведення. Let τ be an idempotent of the semigroup $(\mathrm{Part}_n, *_\rho)$. Then

$$\tau = \tau *_\rho \tau = \tau \wedge \rho \wedge \tau = \tau \wedge \rho, \quad \text{that is} \quad \tau \leqslant \rho.$$

Conversely, consider $\tau \leqslant \rho$. Thus $\tau = \tau \wedge \rho = \tau \wedge \rho \wedge \tau = \tau *_\rho \tau$. Hence τ is an idempotent of the semigroup $(\mathrm{Part}_n, *_\rho)$. □

Automorphisms of variants of the semigroup Part_n

Theorem 1. *For each automorphism* $\varphi \in \mathrm{Aut}(\mathrm{Part}_n, *_\rho)$ *the lower cone* $[0, \rho]$ *is an invariant subset and the restriction of* φ *on* $[0, \rho]$ *is an automorphism of* $[0, \rho]$ *as a partially ordered set.*

Доведення. Note, that $\varphi(\rho) \leqslant \rho$. Thus by the rules of partial order for the lattice we got that $\varphi(\rho) = \varphi(\rho \wedge \rho \wedge \rho) = \varphi(\rho *_\rho \rho) = \varphi(\rho) *_\rho \varphi(\rho) = \varphi(\rho) \wedge \rho \wedge \varphi(\rho) = \rho \wedge \varphi(\rho)$. Hence $\varphi(\rho) \leqslant \rho$.

Further we show that $[0, \rho]$ is an invariant set for φ. If $x \leqslant \rho$, then $x = x \wedge \rho \wedge \rho = x *_\rho \rho$. Therefore $\varphi(x) = \varphi(x *_\rho \rho) = \varphi(x) *_\rho \varphi(\rho) = \varphi(x) \wedge \rho \wedge \varphi(\rho) = \varphi(x) \wedge \varphi(\rho) \leqslant \varphi(\rho) \leqslant \rho$.

If $x \leqslant y \leqslant \rho$, then $x = x \wedge \rho \wedge y = x *_\rho y$. Thus

$$\varphi(x) = \varphi(x) *_\rho \varphi(y) = \varphi(x) \wedge \rho \wedge \varphi(y) = \varphi(x) \wedge \varphi(y).$$

Hence $\varphi(x) \leqslant \varphi(y)$ that is φ saves the partial order on $[0, \rho]$.

Thus the restriction $\varphi|_{[0,\rho]} : [0, \rho] \to [0, \rho]$ is a homomorphism of partially ordered sets. Since inverse mapping φ^{-1} belongs to $\mathrm{Aut}(L, *_a)$, then $\varphi|_{[0,\rho]}$ is bijective, that is $\varphi|_{[0,\rho]}$ is an automorphism of $[0, \rho]$ as partially ordered set. □

Corollary 1. *For each automorphism* $\varphi \in \mathrm{Aut}(\mathrm{Part}_n, *_\rho)$ *holds equality* $\varphi(\rho) = \rho$.

Theorem 2. $\mathrm{Aut}\,\mathrm{Part}_n \simeq S_n$.

Доведення. Схема доведення. For each permutation $\pi \in S_n$ we consider the mapping φ_π such that it maps the partition $a_1 \dots a_k | \dots | a_r \dots a_n$ to the partition $a_1^\pi \dots a_k^\pi | \dots | a_r^\pi \dots a_n^\pi$. Then the mapping φ_π is an automorphism of the lattice Part_n.

Conversely, consider $\psi \in \mathrm{Aut}\,\mathrm{Part}_n$. The lower cone of the element $\rho \in \mathrm{Part}_n$ is isomorphic to the lattice Part_{n-1} if and only if ρ have the form $a | \mathsf{N} \setminus \{a\}$. Thus ψ maps the partition $a | \mathsf{N} \setminus \{a\}$

to the partition $b|\mathsf{N} \setminus \{b\}$ of the same form. By such property for the automorphism ψ we naturally have the permutation $\pi_\psi \in S_n$ such that π_ψ maps a to b. The automorphism $\psi\varphi_{\pi_\psi}^{-1}$ leaves still each partition $a|\mathsf{N} \setminus \{a\}$. The partition $\tau = a|b|\mathsf{N} \setminus \{a,b\}$ is the only one which lays in lower cones of both partitions $\mu = a|\mathsf{N} \setminus \{a\}$ and $\nu = b|\mathsf{N} \setminus \{b\}$ and have a lower cone isomorphic to Part_{n-2}. Since automorphism $\psi\varphi_{\pi_\psi}^{-1}$ leaves still both partitions μ and ν, then it leaves still the partition τ.

Similarly it can be proved that $\psi\varphi_{\pi_\psi}^{-1}$ leaves still each partition of the form $a_1|a_2|\ldots|a_k|\mathsf{N} \setminus \{a_1, a_2, \ldots, a_k\}$ with only unique one-element block. Hence $\psi\varphi_{\pi_\psi}^{-1}$ leaves still all partitions. Thus $\psi = \varphi_{\pi_\psi}$. □

A partition τ has a *type* $(l_1, l_2, \ldots, l_n)$ if it contains l_1 blocks of cardinality 1, l_2 blocks of cardinality 2, l_n blocks of cardinality n. Obviously, that $l_1 + 2l_2 + \cdots + nl_n = n$.

Theorem 3. *Let τ be a partition of type $\langle l_1, l_2, \ldots, l_n \rangle$. Then automorphism group of the interval $[0, \tau]$ is isomorphic to the direct product $G_1 \times G_2 \times \cdots \times G_n$ of wreaths products $G_k = S_{l_k} \wr S_k$.*

Доведення. Схема доведення. The interval $[0, \tau]$ is isomorphic to the direct product of lattices of partitions
$\underbrace{\mathrm{Part}_1 \times \cdots \times \mathrm{Part}_1}_{l_1} \times \underbrace{\mathrm{Part}_2 \times \cdots \times \mathrm{Part}_2}_{l_2} \times \cdots \times \underbrace{\mathrm{Part}_n \times \cdots \times \mathrm{Part}_n}_{l_n}$.

We will show that isomorphism φ of the interval $[0, \tau]$ permute the same type multipliers of this product (in the wreath product for given k it corresponds to the multiplier S_{l_k}) and in each multiplier φ acts the automorphism of the corresponding lattice (in the wreath product it gives a multiplier S_k).

Obviously, that each element from $G_1 \times G_2 \times \cdots \times G_n$ is an automorphism of the interval $[0, \tau]$. In the proof of the fact that there is no other automorphisms not obvious only the part related to the permutation of multipliers. It can be proven in the next way: we pick k such that it is the biggest number for which $l_k \neq 0$ and one of multipliers Part_k. The identity of this multiplier is an element of rank k with the lower cone isomorphic to Part_k. Thus the identity is mapped to the element with the same properties and so cone of the identity goes forward... Hence a given multiplier Part_k is mapped to some element of the form Part_k, thus automorphism permute multipliers of the form Part_k. Also the part $\underbrace{\mathrm{Part}_k \times \cdots \times \mathrm{Part}_k}_{l_k}$ is an invariant.

Further we move to the smaller numbers k and use the same scheme of proof.

□

Corollary 2. *Automorphism of the interval $[0, \tau]$ save the type of the partition.*

Let A be a disjoined union $A = A_1 \cup \cdots \cup A_k$ of the sets $A_1, \ldots, A_k$ of powers $m_1, \ldots, m_k$ respectively. By $n_r(m_1, \ldots, m_k)$ we denote the number of partitions of the set A on r blocks, such that each block contains not more than one element from each of sets $A_1, \ldots, A_k$.

By $[x]_k$ we denote falling factorial number k, such that $[x]_k = x(x-1)\cdots(x-k+1)$.

Proposition 2. *Two next equalities are met.*

a) $n_r(m) = \begin{cases} 1, & \text{якщо } m = r; \\ 0, & \text{в інших випадках.} \end{cases}$

b) $n_r(m_1, \ldots, m_{i+1}) = \sum\limits_{j=0}^{m_{i+1}} \binom{m_{i+1}}{j} n_{r+j-m_i} \cdot$
$\cdot (m_1, \ldots, m_i)[r + j - m_i]_j$.

Доведення. a) Obvious, because all blocks in this case are one-element (singletons) blocks.

b) We group all partitions of the set $A_1 \cup \cdots \cup A_i \cup A_{i+1}$ in classes by the number of singletons which belong to the set A_{i+1}. If there is j elements with this property, then they can be picked in $\binom{m_{i+1}}{j}$ ways. Other $m_{i+1} - j$ elements we join to the blocks of the partition of the set $A_1 \cup \cdots \cup A_i$ on $r + j - m_i$ blocks. This partition can be obtained in $n_{r+j-m_i}(m_1, \ldots, m_i)$ ways. Since different elements must be joined to different blocks it can be done in $[r + j - m_i]_j$ ways. □

Denote the number $\sum\limits_{r \leqslant m_1 + \cdots + m_k} n_r(m_1, \ldots, m_k)$ by $t(m_1, \ldots, m_k)$.

Proposition 3. *Let the partition $\rho \in \mathrm{Part}_n$ have a form*

$$\mathsf{N} = A_1 \cup \cdots \cup A_p \cup B_1 \cup \cdots \cup B_q,$$

and the partition $\tau \in [0, \rho]$ have the form

$$\mathsf{N} = C_1 \cup \cdots \cup C_{m_1 + \cdots + m_p} \cup B_1 \cup \cdots \cup B_q,$$

where each block $A_1, \ldots, A_r$ by the transition from ρ to τ is granulated on $m_1, \ldots, m_p$ smaller blocks. Then the set

$$M_\tau = \{\mu \in \mathrm{Part}_n \mid \mu \wedge \rho = \tau\}$$

contains

$$R_\tau = \sum_{r \leqslant m_1 + \cdots + m_k} \sum_{k=1}^{q} n_r(m_1, \ldots, m_p) S(q,k)(r+1)^k$$

elements, where $S(q,k)$ is a Stirling number of the second kind.

Доведення. The partition $\mu \wedge \rho$ contains a block B_i if and only if B_i belong to one of the blocks of the partition μ. There is the same for blocks of C_j. Furthermore, if different blocks C_{j_1} and C_{j_2} belong to the same block of the partition ρ, then they belong to the different blocks of the partition μ.

Thus partition μ belongs to the set M_τ if and only if it can be obtained by the next rules. Firstly, by unifying blocks $C_1, \ldots, C_{m_1+\cdots+m_p}$ we get r blocks of the partition μ, such that non of two different blocks C_{j_1} and C_{j_2} which belong to the one block of the partition ρ do not belong to the same block of the partition μ. It can be done in $n_r(m_1, \ldots, m_p)$ ways. Secondly, by the unifying blocks $B_1, \ldots, B_q$ we get k more blocks. It can be done in $S(q,k)$ ways. Each of these k blocks we do not change or join to one of r blocks, such that we get on the first step. It can be done in $(r+1)^k$ ways.

Finally, we count the sum by all possible r and k this gives us a power of the set M_τ. □

For the formulation of the main result about the automorphism group of the variant $(\mathcal{L}(V), *_A)$ we need the next construction.

Let (G, M) be a permutation group, $O_1, \ldots, O_k$ are orbits of this group and $(H_1, N_1), \ldots, (H_k, N_k)$ are permutation groups. The permutation group, such that elements of this group are sets $(g; f_1, \ldots, f_k)$ where g is an elements G and for each i $1 \leqslant i \leqslant k$, f_i is a function from O_i to H_i is called *generalized wreath product.* The group $G \wr (H_1, \ldots, H_k)$ acts on the set $\bigcup_{i=1}^k (O_i \times N_i)$ by the rule

$$(m, n)^{(g; f_1, \ldots, f_k)} = (m^g, n^{f_i(m)}), \quad \text{if } m \in N_i.$$

Theorem 4. *Let* $(\mathrm{Part}_n, \wedge)$ *be a partition semigroup of the n-element set, $\rho \neq 1$ a partition of the type $\langle l_1, l_2, \ldots, l_n \rangle$ from* Part_n, *$\mathcal{O}_1, \ldots, \mathcal{O}_v$ are orbits of automorphism group of the interval $[0, \rho]$, τ_i a representative of the orbit $\mathcal{O}_i$ $(i = 1, 2, \ldots, v)$. Then automorphism group of the variant* $(\mathrm{Part}_n, *_\rho)$ *isomorphic to the generalized wreath product $S = \big((S_{l_1} \wr S_1) \times \cdots \times (S_{l_n} \wr S_n)\big) \wr (S_{R_{\tau_1}-1}, S_{R_{\tau_2}-1}, \ldots, S_{R_{\tau_v}-1})$, where numbers R_{τ_i} are defined by the Proposition 3 and by S_m we denote a symmetric group of the power m.*

Доведення. The set Part_n is a disjoined union of sets $\tau \in [0, \rho]$. Further, consider elements $\mu \in \mathrm{Part}_n$ as an ordered pairs of the form $(\mu \wedge \rho, \mu)$.

Note that the set M_τ consists of the partition τ, such that belongs to the interval $[0, \rho]$ and the "tail"$M_\tau \setminus \{\tau\}$ of partitions, such that lay out of this interval.

By the theorem 1 the restriction of the automorphism $\varphi \in \mathrm{Aut}(\mathrm{Part}_n, *_\rho)$ on $[0, \rho]$ ia an automorphism of $[0, \rho]$. Additionally, by the theorem 3 automorphism group G of the interval $[0, \rho]$ ia isomorphic to the direct product $G_1 \times G_2 \times \cdots \times G_n$ of the wreaths products $G_k = S_{l_k} \wr S_k$.

Consider $\varphi \in \mathrm{Aut}(\mathrm{Part}_n, *_\rho)$ and $\mu \in M_\tau$, thus $\mu \wedge \rho = \tau$. Then

$$\varphi(\tau) = \varphi(\mu \wedge \rho) = \varphi(\mu \wedge \rho \wedge \rho) = \varphi(\mu *_\rho \rho) = \\ = \varphi(\mu) *_\rho \varphi(\rho) = \varphi(\mu) *_\rho \rho = \varphi(\mu) \wedge \rho,$$

hence $\varphi(\mu) \in M_{\varphi(\tau)}$. Since a mapping φ^{-1} exists, it follows that φ induces a bijection M_τ on $M_{\varphi(\tau)}$. Partitions τ and $\varphi(\tau)$ belongs to the interval $[0, \rho]$, then φ induces a bijection of the "tail"$M_\tau \setminus \{\tau\}$ on the "tail"$M_{\varphi(\tau)} \setminus \{\varphi(\tau)\}$. It proves that $\mathrm{Aut}(\mathrm{Part}_n, *_\rho) \subseteq S$.

Conversely, consider $\psi = (g; f_1, \ldots, f_v) \in S$. By the definition of S, follows that restriction of ψ on $[0, \rho]$ is an automorphism of $[0, \rho]$. Further it can be directly checked that the mapping

$$(\mu \wedge \rho, \mu) \mapsto \big((\mu \wedge \rho)^g, (\mu)^{f_i(\mu \wedge \rho)}\big), \quad \mu \wedge \rho \in \mathcal{O}_i,$$

is an automorphism $(\mathrm{Part}_n, *_\rho)$. It proves the converse statement. □

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