

UDC 519.2

DOI: <https://doi.org/10.17721/1812-5409.2026/1.11>

Javier M. HUARCA, PhD, Prof.

ORCID ID: 0000-0002-1564-2343

e-mail: jhuarcao@usmp.pe

Universidad de San Martín de Porres, Lima, Peru

THE H PROBABILITY DISTRIBUTIONS

Probability distributions are important models in all scientific fields; however, they are essential in statistics. We provide a new class of mixed joint probability distributions with probability spaces that are sequences of nested piled $2D$ -lattices. The methodology for producing these models involves embedding a discrete variable extended β -Benford's distribution into a continuous variable modified Bernstein polynomial. β is the parameter of this family of distributions and denotes the base of an arbitrary number system. Our purpose was to describe the mathematical structure and statistical features behind the interactions of these probability distributions. On the basis of the theoretical development and its results, we conclude that these models are valuable for numerous applications in various scientific disciplines. A concrete decimal system case is introduced to support our hypothesis.

Key words: extended β -Benford distribution, Bernstein polynomials, number system, universal properties of a probability distribution, homotopic simplicial complex.

AMS 2020 classification: 60E05, 93A30; 62R40.

Introduction

Probability distributions have played an important role in all fields of science, especially statistics, since the sixteenth century, initiated by G. Cardano, followed by P. de Fermat and B. Pascal in the seventeenth century; this tradition is carried on to this day by scientists such as (Franklin, 2015; Debnath, & Basu, 2015). For example, after the discovery of the renowned Benford's distribution in 1938 (Benford, 1938), which originally went back to 1881, as surmised by astronomer (Newcomb, 1881). Since then, many papers and books supporting its theoretical and practical applications have been published in various fields of science. Namely, in statistics, mathematics, in business (accounting, economics, finance, and management), physics, chemistry, social science, computer science, etc. This means that there is rich and abundant literature on Benford's distribution.

Some researchers aimed to find generalizations of Benford's law, achieving fruitful results, such as (Fu, Shi, & Su, 2007), who proposed an interesting, first-digit, generalized Benford's law

$$p(x) = N \log_{10} \left(1 + \frac{1}{s + x^q} \right), \quad x = 1, 2, 3, 4, 5, 6, 7, 8, 9,$$

where N is a normalization factor that makes $p(x)$ to be a probability distribution, and s and q are some model parameters, see (Fu, Shi, & Su, 2007) for details. In addition, (Pietronero et al., 2001) showed that Benford's law can be generalized to a series of numbers generated from more complex systems, such as catalogs of seismic activity. (Barabesi, & Pratelli, 2020) provide some properties of the Generalized Benford Law, a flexible model for the distribution of significant digits, and abundant references in the introduction of their publication. Although there is abundant literature on this topic, it remains insufficient. This definition can be expanded to encompass broader domains. In addition, new probability distributions need to be identified, and their applicability extended beyond the current distributions to both classical and quantum fields. This need was one of the motivations for writing this paper.

In the introduction, we begin with a brief presentation of the well-known standard Benford probability distribution (SBPD) for a discrete random variable W and for any positive integer $\beta \geq 2$, defined by

$$B(w; \beta) = \log_{\beta} \left(1 + \frac{1}{w} \right), \quad w = 1, 2, \dots, \beta - 1, \quad (1)$$

where $\sum_{w=1}^{\beta-1} B(w; \beta) = 1$ and $B(w; \beta)$ is non-negative for all w , see the proof of (5). Note that for the decimal system, $\beta = 10$, w in $\{1, 2, 3, \dots, 9\}$ is a special case that our brain understands the most. We derive the expected value, variance, and moment generating function of this SBPD for any number of systems of base $\beta \geq 2$, and obtain the following:

$$E(w; \beta) = \log_{\beta} \left[\frac{\beta^{(\beta-1)}}{(\beta-1)!} \right], \quad (2)$$

$$Var(w; \beta) = \log_{\beta} \left[\frac{\beta^{(\beta-1)^2}}{\prod_{i=2}^{\beta} (i-1)^{2i-3}} \right] - \left[\log_{\beta} \left(\frac{\beta^{(\beta-1)}}{(\beta-1)!} \right) \right]^2, \quad (3)$$

$$M_W(t) = \log_{\beta} \left[\frac{\beta^{e^{(\beta-1)t}}}{\prod_{i=2}^{\beta} (i-1)^{e^{(i-1)t}} - e^{(i-2)t}} \right], \quad (4)$$

where e denotes the Euler number. We note that these impressive results depend only on the parameter β , which is the basis of an arbitrary number system. Namely, if we consider the binary number system then $\beta = 2$, and $E(w; 2) = 1$, $Var(w; 2) = 0$, $M_W(t) = e^t$ for real t .

In Section 1 we demonstrate the properties of the extended SBPD, defined in Equation (5), as a more general case to clarify the veracity of Equations (2), (3), and (4). To the best of our knowledge, we do not know whether these interesting results have been published in the literature.

Henceforth, we establish that parameter β functions as both the base of the numerical system and the base of the logarithm. Any system and logarithm base chosen are automatically convertible by computers to either a natural logarithm or a decimal logarithm during processing.

In this study, we provide a novel family of mixed joint probability distributions, denoted by $\mathcal{H}(x, w; \beta, j)$, associated with an extended β -Benford distribution, Equation (5), and special polynomials formulated for an arbitrary number system of bases $\beta \geq 2$. We show that each member of this family defines a mixed two-dimensional probability distribution with its own features and characteristics. We then provide a concrete example for $\beta = 10$ in the decimal system to clarify the theory.

In other words, the primary goal of this research is to introduce finite families of mixed probability distributions characterized by their nested-piled 2D-lattices and their features. It is organized as follows: Section 1 provides the proposed extended β -Benford probability distribution and its preliminary features and methodology. The crucial theoretical developments for a generalized $\mathcal{H}_G(x, w; \beta, j)$ distribution are formalized in Section 2 starting with Theorem 2 and its generalization Theorem 3, followed by several propositions. To clarify this theory, we choose the decimal system $\beta = 10$ as an application. All other bases work in the same manner. Section 4 discusses some theoretical and practical implications of the main $\mathcal{H}$ findings and concludes with a brief summary of the contributions of this study.

This study makes four contributions. First, a new generalized β -Benford distribution for all digits of any number system of bases β was formulated. This distribution constitutes one of the two key components of the structure of our models. Second, a family of 2-dimensional mixed joint probability distributions is introduced along with its statistical features. These are the models called the $\mathcal{H}$ distributions. Third, a connection between these $\mathcal{H}$ distributions and Bézier curves of degree $\beta - 1$ with β control points is provided. Fourth, an analytic allocation distribution function is established.

1. Material and Methodology

The Equations (2), (3), and (4) are the fundamental properties of the SBPD. This widely acknowledged SBPD is also referred to as the first non-zero digit law or the distribution of the most significant digits, as it is restricted to sets of data with non-zero first digits. This means that the digit zero of a number system is not an element of its domain. However, there are many sets of data in all fields of science whose leftmost digit begins with zero, namely: SSN, student ID, ORCID, bank accounts and loans, constants in physics, etc. Therefore, to examine the frequency of the first positional digit for this kind of data, we need to extend the domain in (1) of the SBPD to all digits of any number system.

Thus, to remedy these shortcomings, we provide an extended Benford probability distribution (EBPD), which includes all basic symbols of an arbitrary number system of base β , in the following theorem.

Theorem 1. Let W be a discrete random variable. W has an EBPD $\mathcal{H}(w; \beta)$ given by

$$\mathcal{H}(w; \beta) = \frac{1}{\log_{\beta}(\beta + 1)} \log_{\beta}\left(1 + \frac{1}{w + 1}\right), \quad w = 0, 1, 2, \dots, \beta - 1, \quad (5)$$

where $\beta \geq 2$ is a positive integer that plays the role of the base of a number system and, at the same time, is the base of the logarithm function, as we established above.

We clearly observe that this discrete variable W includes all the basic values of a numerical system. For example, if $\beta = 2$ or $\beta = 10$, we have the well-known binary ($w = 0, 1$) and decimal ($w = 0, 1, 2, \dots, 9$) systems and $\log_2(\cdot)$ and $\log_{10}(\cdot)$, respectively. It is understood that any logarithm function of base β is related to the natural logarithm $\ln$. Fig. 1 illustrates the shapes of these probability distributions for distinct positive integer values of $\beta \geq 2$. For example, for $\beta = 2$, $w = 0; 1$ with probability values $\mathcal{H}(0; 2) = 0.63$ and $\mathcal{H}(1; 2) = 0.37$. For $\beta = 3$, $w = 0; 1; 2$ with probabilities $\mathcal{H}(w; 3) = 0.5, 0.29, 0.21$ (or 50, 29, 21 %), respectively. Note that the total probability in these two examples, as well as in all other instances, is 1, as required, and decays exponentially.

For each value of β and $w = 0, 1, 2, \dots, \beta - 1$, gluing together consecutive points, we have topological building blocks called simplicial complexes. A 0-dimensional simplex is a vertex; a 1-dimensional simplex is a line segment with two vertices; a 2-dimensional simplex has three vertices and three edges, and so on. We clarify these topological objects a little more in Section 2. See (Singer, & Thorpe, 1967) for more details.

Proof. It is easy to verify that (5), as well as (1), satisfies the two universal properties of a probability distribution (p.d.). That is, $\mathcal{H}(w; \beta) \geq 0$ for all w and $\sum_{w=0}^{\beta-1} \mathcal{H}(w; \beta) = 1$. For the second property, when executing the summation, all terms cancel out each other except for the last term, $\log_{\beta}(\beta + 1)$, which survives, and then it cancels out with the constant of normality of (5). Also, to prove it, one should use the telescoping series. $\square$

Equation (5) is one of the two fundamental pieces in the construction of the distribution $\mathcal{H}_G(x, w; \beta, j)$. In what follows, we write $\log(\cdot)$ to denote $\log_{\beta}(\cdot)$. We now compute some natural properties of (5).

Proposition 1. Let W be a random variable distributed as in (5), then the expected value and the variance are given by

$$E(W; \beta) = \frac{1}{\log(\beta + 1)} \log\left(\frac{(\beta + 1)^{(\beta-1)}}{\beta!}\right), \quad (6)$$

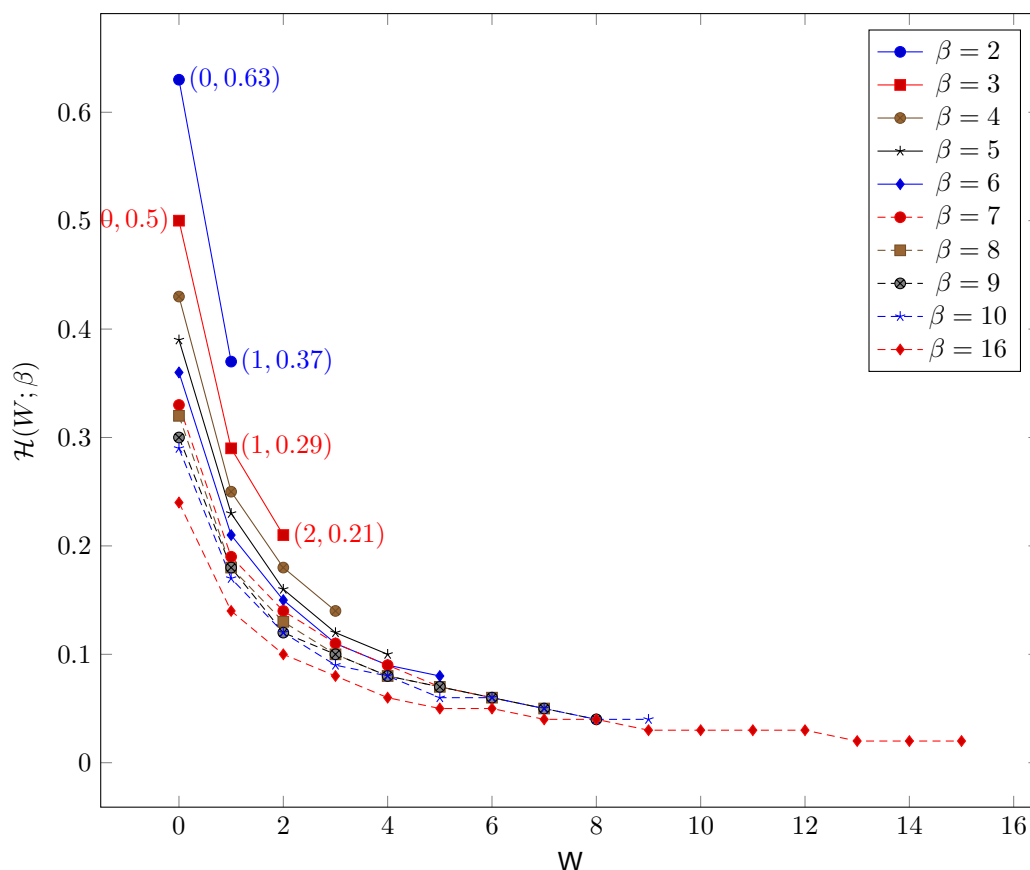


Fig. 1. Graphs of $\mathcal{H}(w; \beta)$ for distinct bases $\beta = 2, 3, \dots, 10, 16$.

$$Var(W; \beta) = \frac{1}{\log(\beta + 1)} \log \left[\frac{(\beta + 1)^{(\beta - 1)^2}}{\prod_{i=1}^{\beta} i^{2i - 3}} \right] - \left[\frac{1}{\log(\beta + 1)} \log \left(\frac{(\beta + 1)^{(\beta - 1)}}{\beta!} \right) \right]^2. \quad (7)$$

Proof. In general, one could show this proposition by finding the k th moment about the origin of the random variable W , usually denoted by μ'_k , the expected value of W^k .

Symbolically, for $\beta \geq 2, w = 0, 1, 2, \dots, \beta - 1$, we have

$$\begin{aligned} E(W^k) &= \frac{1}{\log(\beta + 1)} \sum_{w=0}^{\beta-1} w^k \log \left(1 + \frac{1}{w+1} \right) = \frac{1}{\log(\beta + 1)} \log \left[\prod_{w=0}^{\beta-1} \left(\frac{w+2}{w+1} \right)^{w^k} \right] = \\ &= \frac{1}{\log(\beta + 1)} \log \left[\frac{(\beta + 1)^{(\beta-1)^k}}{\prod_{i=1}^{\beta} i^{(i-1)^k - (i-2)^k}} \right], \end{aligned}$$

where $\mu'_1 = E(W; \beta)$ when $k = 1$ and $Var(W; \beta) = \mu'_2 - (\mu'_1)^2$ when $k = 1, 2$. □

We can notice that as β increases, both the expectation and the variance increase.

Proposition 2. Let W be a random variable whose probability distribution is given in (5), then its moment generating function, for all real values of t is given by

$$M_W(t) = \frac{1}{\log(\beta + 1)} \log \left[\frac{(\beta + 1)^{e^{(\beta-1)t}}}{\prod_{i=1}^{\beta} i^{e^{(i-1)t} - e^{(i-2)t}}} \right], \quad \beta \geq 2, t \in (-\infty, +\infty). \quad (8)$$

Proof. We use the definition of the moment-generating function

$$M_W(t) = E(e^{tW}) = \sum_{w=0}^{\beta-1} e^{tw} \frac{1}{\log(\beta + 1)} \log \left(1 + \frac{1}{w+1} \right) = \frac{1}{\log(\beta + 1)} \log \left[\prod_{w=0}^{\beta-1} \left(\frac{w+2}{w+1} \right)^{e^{tw}} \right].$$

Expanding the product operator, on the right-hand side, associating the exponents of each pair of consecutive factors, and simplifying, yields the proof. □

Of course, from this $M_W(t)$, we can generate all of the moments of W by successively differentiating $M_W(t)$, with respect to t , and then evaluating the result at $t = 0$. This is another way to prove the Proposition 1. We leave this to the reader. Notice that $M_W(0) = 1$, for every $\beta \geq 2$ and $t \in (-\infty, +\infty)$.

2. Main theory and results

In this section, following all notations as above, we formulate the probability distribution $\mathcal{H}(\cdot)$ of two random variables. It is a joint distribution of a mixed random vector (X, W) that is described in the next theorem.

Theorem 2. A random vector (X, W) has a mixed joint probability distribution, denoted by $\mathcal{H}(x, w; \beta)$, with parameter $\beta \geq 2$ if its joint distribution is given by

$$\mathcal{H}(x, w; \beta) = \binom{\beta-1}{w} \frac{x^w (\beta-x)^{\beta-w-1}}{\beta^{\beta-1} \log(\beta+1)} \log\left(1 + \frac{1}{w+1}\right), \quad w = 0, 1, 2, \dots, \beta-1. \quad 0 \leq x \leq \beta. \quad (9)$$

It is a bivariate distribution of a continuous random variable X and a discrete random variable W , sitting on a mesh domain. The foundation of $\mathcal{H}(x, w; \beta)$ is the result of embedding the EBPd defined in (5) with a generalization of β -Bernstein basis polynomials of degree $\beta-1$ over an interval $[0, \beta]$, for any number system of base $\beta \geq 2$. This theorem is the backbone of this research from which abundant theoretical information emerges. This probability distribution has certain properties and characteristics that, traditionally, any probability distribution must have. These include the cumulative distribution, marginals, conditionals, expected values, variance, covariance, correlation coefficient, moment generating function, characteristic function, etc. Our goal in this section is to investigate and characterize the mathematical and statistical structures underlying these probability distributions. Primarily, $\mathcal{H}(x, w; \beta)$ needs to satisfy the universal properties:

$$(i) \quad \mathcal{H}(x, w; \beta) \geq 0 \quad \text{for all values of } (x, w) \quad \text{and} \quad (ii) \quad \sum_{w=0}^{\beta-1} \int_0^\beta \mathcal{H}(x, w; \beta) dx = 1.$$

Proof. The property (i) is clear, since all factors of $\mathcal{H}(x, w; \beta)$ are positive for each value of (x, w) on their domains. For (ii) the key step is to solve the integral $\int x^w (\beta-x)^{\beta-w-1} dx$. To solve it, one may use any mathematical software. Here we use (Maple.Soft, 2024), obtaining

$$\beta^{\beta-w-1} \frac{1}{w+1} x^{w+1} \text{hypergeom}([w+1, -\beta+w+1], [2+w], \frac{x}{\beta}),$$

then, evaluating at the limits of the integral $x = \beta$ and $x = 0$, we have $\beta^\beta \Gamma(\beta-w) \Gamma(w+1) / \Gamma(\beta+1)^1$. Now, replacing this result in the Equation (ii) then expanding the binomial coefficient and simplifying it turns out

$$\sum_{w=0}^{\beta-1} \frac{\log(1 + \frac{1}{w+1})}{\log(\beta+1)} = 1,$$

which is the case of Equation (5) and completes the proof. $\square$

Proposition 3. Let (X, W) be a random vector with joint probability distribution established in Theorem 2 for $\beta \geq 2$, then the marginal probability density function of X and the marginal probability mass function of W are, respectively,

$$\mathcal{H}(x; \beta) = \sum_{w=0}^{\beta-1} \binom{\beta-1}{w} \frac{x^w (\beta-x)^{\beta-w-1}}{\beta^{\beta-1} \log(\beta+1)} \log\left(1 + \frac{1}{w+1}\right), \quad 0 \leq x \leq \beta, \quad (10)$$

$$\mathcal{H}(w; \beta) = \frac{1}{\log(\beta+1)} \log\left(1 + \frac{1}{w+1}\right), \quad w = 0, 1, 2, \dots, \beta-1. \quad (11)$$

Proof. Obviously, to be $\mathcal{H}(x; \beta)$ and $\mathcal{H}(w; \beta)$ probability distribution, by themselves, they must satisfy the universal properties (i): non-negativity and (ii): total probability. In both cases the non-negativity is clear. For the (ii) the Equation (11), $\mathcal{H}(w; \beta) = \frac{1}{\log(\beta+1)} \log(1 + \frac{1}{w+1})$, $w = 0, 1, 2, \dots, \beta-1$, was shown in (5) that $\mathcal{H}(w; \beta)$ satisfies the universal property. That is, $\sum_{w=0}^{\beta-1} \mathcal{H}(w; \beta) = 1$. Thus, it qualifies to be a discrete marginal probability distribution of $\mathcal{H}(x, w; \beta)$. To show that (10) is a probability density function (p.d.f) it can be written as

$$\mathcal{H}(x; \beta) = \sum_{w=0}^{\beta-1} \binom{\beta-1}{w} \frac{x^w (\beta-x)^{\beta-w-1}}{\beta^{\beta-1}} \mathcal{H}(w; \beta), \quad 0 \leq x \leq \beta. \quad (12)$$

Now, integrating both sides, we solve the definite integral $\int_0^\beta \mathcal{H}(x; \beta) dx$ of Equation (12) as we did in the Theorem 2 and simplifying, we get the result $\int_0^\beta \mathcal{H}(x; \beta) dx = 1$. $\square$

¹ $\Gamma(m) = (m-1)!$ is the discrete gamma function.

We show a special case and its applications in the next section. We continue this section showing some standard properties of $\mathcal{H}(x, w; \beta)$.

Proposition 4. Consider all the random variables satisfying assumption of Proposition 1 Then, the means and the variances of X and W are, respectively,

$$E(x; \beta) = \frac{\beta}{(\beta + 1)\log(\beta + 1)} \log\left[\frac{(\beta + 1)^\beta}{\beta!}\right], \quad \beta \geq 2, \quad (13)$$

$$Var(x; \beta) = \frac{\beta^2}{(\beta + 1)(\beta + 2)\log(\beta + 1)} \log\left[\frac{(\beta + 1)^{\beta(\beta+1)}}{\prod_{i=1}^{\beta} i^{2i}}\right] - \left(\frac{\beta}{(\beta + 1)\log(\beta + 1)} \log\left[\frac{(\beta + 1)^\beta}{\beta!}\right]\right)^2. \quad (14)$$

Proof. The expected value $E(w; \beta)$ and the variance $Var(w; \beta)$ have already been proven in Proposition 1. To demonstrate (13) and (14) we first determine the k th moment about the origin of X . Consider Theorem 2 and follow the strategy that we have done in Proposition 1

$$E(X^k) = \sum_{w=0}^{\beta-1} \int_0^\beta x^k \mathcal{H}(x, w; \beta) dx = \sum_{w=0}^{\beta-1} \int_0^\beta \binom{\beta-1}{w} \frac{x^{w+k} (\beta-x)^{\beta-w-1}}{\beta^{\beta-1} \log(\beta+1)} \log\left(1 + \frac{1}{w+1}\right) dx. \quad (15)$$

Solving the integral in (15), with (Maple.Soft, 2024), and simplifying, we obtain

$$\begin{aligned} E(X^k) &= \sum_{w=0}^{\beta-1} \frac{\beta^{k+1} \Gamma(\beta) \Gamma(w+k+1)}{\Gamma(w+1) \Gamma(\beta+k+1)} \frac{\log(1 + \frac{1}{w+1})}{\log(\beta+1)} = \frac{\beta^k \beta!}{(\beta+k)! \log(\beta+1)} \sum_{w=0}^{\beta-1} \log\left[\frac{w+2}{w+1}\right]^{\frac{(w+k)!}{w!}} = \\ &= \frac{\beta^k \beta!}{(\beta+k)! \log(\beta+1)} \log\left[\prod_{w=0}^{\beta-1} \left[\frac{w+2}{w+1}\right]^{\frac{(w+k)!}{w!}}\right] = \theta_k(\beta) \log\left[\frac{(\beta+1)^{\frac{(k+\beta-1)!}{(\beta-1)!}}}{\prod_{i=1}^{\beta} i^{\frac{k(k+i-2)!}{(i-1)!}}}\right], \quad \beta \geq 2, \end{aligned}$$

where $\theta_k(\beta) = \frac{\beta^k \beta!}{(\beta+k)! \log(\beta+1)}$. To obtain the last expression, of the right hand, expand the product for w then find the pattern of the factors and combine their exponents.

Now, from the result of (15), we find $E(X)$, $E(X^2)$ and $Var(X) = E(X^2) - (E(X))^2$ as special cases. That is, for $k = 1$,

$$\theta_1(\beta) = \frac{\beta}{(\beta+1)\log(\beta+1)}$$

and for $k = 2$,

$$\theta_2(\beta) = \frac{\beta^2}{(\beta+1)(\beta+2)\log(\beta+1)}.$$

Thus, the mean value in (13) and the variance in (14) are, respectively,

$$E(x; \beta) = \theta_1(\beta) \log\left[\frac{(\beta+1)^\beta}{\beta!}\right], \quad \beta \geq 2,$$

$$Var(x; \beta) = \theta_2(\beta) \log\left[\frac{(\beta+1)^{\beta(\beta+1)}}{\prod_{i=1}^{\beta} i^{2i}}\right] - (\theta_1(\beta) \log\left[\frac{(\beta+1)^\beta}{\beta!}\right])^2.$$

□

Additional key features of this $\mathcal{H}(x, w; \beta)$ distribution include its covariance and the linear correlation coefficient, which we present in a generalized form in Equations (25) and (26), respectively.

3. Generalization of Theorem 2

Now we present an analogue of Theorem 2 and more general features of the $\mathcal{H}$ -distribution. That is, the Theorem 2, Equation (9), can be generalized over an underlying family of nested $2D$ -lattices $\Omega_\beta(j) = \{w\} \times [0, \beta - j]$ for each $j, w = 0, 1, \dots, \beta - 1$ as follows.

Theorem 3. Consider all the data from Theorem 2 and let $j = 0, 1, \dots, \beta - 1$, then the generalized version of Theorem 2 is given by

$$\mathcal{H}_G(x, w; \beta, j) = \binom{\beta-1}{w} \frac{\beta x^w (\beta-j-x)^{\beta-w-1}}{(\beta-j)^\beta} \frac{\log(\frac{w+2}{w+1})}{\log(\beta+1)}, \quad w = 0, 1, \dots, \beta-1, 0 \leq x \leq \beta-j. \quad (16)$$

For any number system of base $\beta \geq 2$ and each fixed j there is a generalized mixed probability distribution, denoted by $\mathcal{H}_G(\cdot)$, defined in a sample space $\Omega_\beta(j) = \{(x, w) : x \in [0, \beta - j], w = 0, 1, 2, \dots, \beta - 1\}$. In other words, given β , for each $j \in \{0, 1, 2, \dots, \beta - 1\}$ there is a probability distribution $\mathcal{H}_G(x, w; \beta, j)$ which is a homogeneous polynomial of degree $\beta - 1$ with β monomial terms, one term for each value of w . See Tab. 1 of Section 4 and see Fig. 2 for $\beta = 10$. Note that when $j = 0$, Theorem 2 shows up with all its features discussed above. Also, when $j = \beta - 1$ this generalization $\mathcal{H}_G(\cdot)$ becomes a standard Bernstein polynomial of degree $\beta - 1$ defined on the interval $[0, 1]$ or Bézier curves with β control points given by the marginal discrete probability distribution $\mathcal{H}_G(W; \beta, j)$ of Equation (19) and the curves are given by the marginal density probability distribution $\mathcal{H}_G(X, \beta, j)$ of Equation (18).

We have found, from this generalization, sets of families of probability distributions, one family for each fixed base β , with β members in the family. It is well-known that there are infinitely many number systems. Hence, we have infinitely many families of probability distributions available. The news is good!. For example, for the family called the decimal system $\beta = 10$, there are 10 probability distributions, one for each j . Thus, with $j = 0, 1, 2, \dots, 9$, each probability distribution has 10 basis polynomials of degree 9 each, one monomial term for each w with $w = 0, 1, 2, \dots, 9$. So, this example gives us 10^2 basis polynomials. See Tab. 1. We elaborate an example in the next section. All these probability distributions of number systems are sitting over probability spaces $(\Omega_\beta, \mathcal{F}_\beta, \mathbb{P}_\beta)$ of descending sequence of nested $2D$ -lattices (or filtrations)

$$\{w\} \times [0, \beta] \supset \{w\} \times [0, \beta - 1] \supset \dots \supset \{w\} \times [0, \beta - j] \supset \dots \supset \{w\} \times [0, 1], \quad w = 0, 1, \dots, \beta - 1$$

for each β and each $j = 0, 1, 2, \dots, \beta - 1$, respectively. These sequences are dependent on the parameter β of the number system. We point out that these families of sets of descending chains are homotopic topological spaces that hold some interesting properties of topological invariance. We briefly explain these properties in the next section.

Proof. The Theorem 3, Equation (16), must satisfy the universal properties of a probability distribution. That is,

$$(i) \mathcal{H}_G(x, w; \beta, j) \geq 0, \text{ for all } (x, w),$$

$$(ii) \sum_{w=0}^{\beta-1} \int_0^{\beta-j} \binom{\beta-1}{w} \frac{\beta x^w (\beta - j - x)^{\beta-w-1}}{(\beta - j)^\beta} \frac{\log(1 + \frac{1}{w+1})}{\log(\beta + 1)} dx = 1. \quad (17)$$

(i) is obvious. To show (ii), one may proceed as in Theorem 2. We first solve the integral $\int x^w (\beta - j - x)^{\beta-w-1} dx$ which results

$$\frac{(\beta - j)^{\beta-w-1} x^{w+1}}{w+1} \text{hypergeom}([w+1, -\beta+w+1], [w+2], \frac{x}{\beta-j}).$$

Then we evaluate in the limits of the integral $x = \beta - j$ and $x = 0$, then simplifying it results $\frac{(\beta-j)^\beta \Gamma(\beta-w) \Gamma(w+1)}{\Gamma(\beta+1)}$.

Now, replace this result into (17), then run the sum and simplify to get 1. $\square$

Same to Proposition 3, for any base $\beta \geq 2$, the marginal distributions for X and W , respectively, are given by

$$\mathcal{H}_G(x; \beta, j) = \sum_{w=0}^{\beta-1} \binom{\beta-1}{w} \frac{\beta x^w (\beta - j - x)^{\beta-w-1}}{(\beta - j)^\beta \log(\beta + 1)} \log(1 + \frac{1}{w+1}), \quad 0 \leq x \leq \beta - j, \quad j = 0, \dots, \beta - 1, \quad (18)$$

$$\mathcal{H}_G(w; \beta, j) = \frac{1}{\log(\beta + 1)} \log(1 + \frac{1}{w+1}), \quad w = 0, 1, 2, \dots, \beta - 1. \quad (19)$$

Equations (5) and (11) already explain (19). To clarify (18), let's write it as follows

$$\mathcal{H}_G(x; \beta, j) = \sum_{w=0}^{\beta-1} \binom{\beta-1}{w} \frac{\beta x^w (\beta - j - x)^{\beta-w-1}}{(\beta - j)^\beta} \psi_\beta(w), \quad 0 \leq x \leq \beta - j, \quad (20)$$

where $\psi_\beta(w)$ is discussed in Proposition 3. With the expression (20) we have found a family of generalization of Bézier curves of degree $\beta - 1$ with β control points given by $\psi_\beta(w)$ for each w satisfying the condition $\sum_{w=0}^{\beta-1} \psi_\beta(w) = 1$. But, in this case, these expressions are families of probability distributions. Furthermore, the equations

$$\mathcal{B}(x, w; \beta, j) = \binom{\beta-1}{w} \frac{\beta x^w (\beta - j - x)^{\beta-w-1}}{(\beta - j)^\beta}, \quad j, w = 0, 1, 2, \dots, \beta - 1 \quad (21)$$

are, for each fixed j in $\{0, 1, 2, 3, \dots, \beta - 1\}$, the well-known Bernstein basis polynomials in a general form that we have extended over an arbitrary real lattice $\{w\} \times [0, \beta - j]$ which includes the interval $\{w\} \times [0, 1]$ as a special case. *Claim:* All theories developed about Bernstein polynomials, Bézier curves, splines, binomial distribution, and relations to Gamma, Beta, Hypergeometric, etc. and their applications are also applicable for this new distribution of probabilities $\mathcal{H}_G(X, W; \beta, j)$. Explicitly, the distribution in (20) seen from the Bernstein basis polynomial point of view of degree $\beta - 1$ forms a *partition of unity*, as we tested above. Subsequently, we present propositions that delineate the fundamental statistical characteristics of the generalized distribution $\mathcal{H}_G(X, W; \beta, j)$ as defined in (16).

Proposition 5. Let $\mathcal{H}_G(X, W; \beta, j)$ be a generalized p.d. of the random vector (X, W) defined in (16) with marginal distributions given in (18) and (19), respectively. The mean and variance of the random variable X from (18) are given by

$$E(X; \beta, j) = \frac{\beta - j}{(\beta + 1) \log(\beta + 1)} \log\left[\frac{(\beta + 1)^\beta}{\beta!}\right], \quad j = 0, 1, 2, \dots, \beta - 1, \quad (22)$$

$$\text{Var}(X; \beta, j) = \frac{(\beta - j)^2}{(\beta + 1)(\beta + 2) \log(\beta + 1)} \log\left[\frac{(\beta + 1)^{\beta(\beta+1)}}{\prod_{i=1}^{\beta} i^{2i}}\right] - \left(\frac{\beta - j}{(\beta + 1) \log(\beta + 1)} \log\left[\frac{(\beta + 1)^\beta}{\beta!}\right]\right)^2. \quad (23)$$

Proof. At this point, it is worthwhile to notice that the random variable W is immune to this generalization, however the random variable X is infected, because of the changes in its domain for each value of j . It means that all properties of

$\mathcal{H}(w; \beta)$, inclusive the $E(W^k)$ for $k = 1, 2, 3, \dots$ and variance, remains as in Proposition 1. To prove (22) and (23) we just need to find the k th moment of the marginal p.d. $\mathcal{H}(x; \beta, j)$ given in (18), and proceed as in the case of Proposition 4.

$$E(X^k) = \int_0^{\beta-j} X^k \mathcal{H}_G(x; \beta, j) dx = \int_0^{\beta-j} \sum_{w=0}^{\beta-1} \binom{\beta-1}{w} \frac{\beta x^{k+w} (\beta-j-x)^{\beta-w-1}}{(\beta-j)^\beta \log(\beta+1)} \log\left(\frac{w+2}{w+1}\right) dx.$$

Assuming convergence, first, we solve the integral to get

$$E(X^k) = \frac{(\beta-j)^k \beta!}{(\beta+k)! \log(\beta+1)} \sum_{w=0}^{\beta-1} \frac{(k+w)!}{w!} \log\left(\frac{w+2}{w+1}\right) = \theta_k(\beta, j) \log\left[\frac{(\beta+1)^{\frac{(\beta+k-1)!}{(\beta-1)!}}}{\prod_{i=1}^{\beta} i^{(i+k-2)! \frac{k}{(i-1)!}}}\right], \quad j = 0, 1, \dots, \beta-1, \quad (24)$$

where $\theta_k(\beta, j) = \frac{(\beta-j)^k \beta!}{(\beta+k)! \log(\beta+1)}$ and $k = 1, 2, 3, \dots$

Thus, from this result in (24), we get $E(X; \beta, j)$ for $k = 1$, and $E(X^2)$ for $k = 2$ and $Var(X, \beta, j)$, respectively. $\square$

We clearly observe that Equations (13) and (14) of Proposition 4 emerge from Equations (22) and (23) when j is 0.

Proposition 6. Let $\mathcal{H}_G(x, w; \beta, j)$ be a joint mixed p.d. defined in (16) with $\beta \geq 2$. Then, for $j = 0, 1, \dots, \beta-1$, the covariance and correlation coefficient² between X and W are given, respectively, by

$$Cov(X, W; \beta, j) = \frac{(\beta-j) \log\left[\frac{(\beta+1)^{(\beta-1)\beta}}{\prod_{i=1}^{\beta} i^{(2i-2)}}\right]}{(\beta+1) \log(\beta+1)} - \frac{(\beta-j) \log\left[\frac{(\beta+1)^\beta}{\beta!}\right] \log\left[\frac{(\beta+1)^{(\beta-1)}}{\beta!}\right]}{(\beta+1) (\log(\beta+1))^2}, \quad (25)$$

$$Corr(X, W; \beta, j) = \frac{Cov(X, W; \beta, j)}{\sqrt{\left[\frac{(\beta-j)^2 \log\left[\frac{(\beta+1)^{\beta(\beta+1)}}{\prod_{i=1}^{\beta} i^{2i}}\right]}{(\beta+1)(\beta+2) \log(\beta+1)} - \frac{(\beta-j)^2 (\log\left[\frac{(\beta+1)^\beta}{\beta!}\right])^2}{(\beta+1)^2 (\log(\beta+1))^2}\right] \left[\frac{\log\left[\frac{(\beta+1)^{(\beta-1)^2}}{\prod_{i=1}^{\beta} i^{(2i-3)}}\right]}{\log(\beta+1)} - \frac{(\log\left[\frac{(\beta+1)^{(\beta-1)}}{\beta!}\right])^2}{(\log(\beta+1))^2}\right]}}. \quad (26)$$

Proof. The calculations for $E(W)$, $Var(W)$ in Equations (6) and (7), as well as $E(X)$, $Var(X)$ in Equations (22) and (23), and $E(XW)$, combined with substitutions in the standard formulas for covariance and correlation coefficients, make up the proof. See the complete proof in the Appendix 4: proofs. $\square$

We have validated the findings with Maple 2024. Note that the effect of replacing Equation (25) in the numerator of Equation (26) produces the cancellation of the factors $(\beta-j)$ in Equation (26). It means that $Corr(X, W; \beta, j)$ depends only on parameter β . That is to say, the relationship between all members of a family β is the same. See details in Appendix 4, Equation 39. In other words, for each given β the β -distributions, for $j = 0, 1, 2, \dots, \beta-1$, have the same correlation coefficient. As it could be, in practice, all members of a family should satisfy the same properties. We illustrate the decimal system in the Section 4.

Finally, but not least, we present the marginal characteristic functions of $\mathcal{H}_G(x, w; \beta, j)$ as an alternative to their cumulative probability distribution functions.

Proposition 7. Let $\mathcal{H}_G(x, w; \beta, j)$ be a generalized p.d. defined in (16) with marginals distributions given in (18) and (19). Then the characteristic function $\varphi: \mathbb{R} \mapsto \mathbb{C}$ are, respectively, given by

$$\varphi_X(t) = E(e^{itX}) = \sum_{w=0}^{\beta-1} {}_1F1[w+1, \beta+1, i(\beta-j)t] \mathcal{H}_G(w; \beta), \quad t \in \mathbb{R}, \quad (27)$$

$$\varphi_W(t) = E(e^{itW}) = \frac{1}{\log(\beta+1)} \log\left[\frac{(\beta+1)^{e^{(\beta-1)it}}}{\prod_{r=2}^{\beta} r^{e^{(r-1)it} - e^{(r-2)it}}}\right]. \quad (28)$$

where i is the complex imaginary unit and $t \in \mathbb{R}$ is the argument of the characteristic function. $\mathcal{H}_G(w, \beta)$ is given in (19), (11) and (5). ${}_1F1[w+1, \beta+1, i(\beta-j)t]$ is the Kummer confluent hypergeometric function, see (Wolfram.Soft, 2025). All other notations are already presented above.

Proof. The proof of the characteristic function given in Equation (27) follows by solving the definite integral $\int_0^{\beta-j} e^{itx} \beta x^w \times (\beta-j-x)^{\beta-w-1} dx$. On this occasion, (Mathematica.Soft, 2024) is our ally. We have

$$\left(\frac{1}{\beta-j}\right)^{-w} (\beta-j)^{\beta-w} \beta \Gamma(\beta-w) \Gamma(w+1) \text{Hypergeometric1F1Regularized}[w+1, \beta+1, i(\beta-j)t],$$

$$\text{if } Re(\beta) > Re(w) > -1.$$

²We compute the covariance $Cov(X, W)$ and correlation coefficient $Corr(X, W)$ using K. Pearson's product moment because both random variables are quantitative. However, due to the mixed nature of the variables, a point multi-serial correlation coefficient could be considered, but it is not the focus of this study. Interested readers can see (Wiley.StatsRef, 2014; Tate, 1954).

Next, consider the relation $Hypergeometric1F1Regularized[w+1, \beta+1, (\beta-j)it] = \frac{{}_1F1[w+1, \beta+1, (\beta-j)it]}{\Gamma(\beta+1)}$. (Wolfram.Soft, 2025) and the result of the integral to replace in

$$\sum_{w=0}^{\beta-1} \binom{\beta-1}{w} \beta \int_0^{\beta-j} \frac{e^{itx} x^w (\beta-j-x)^{\beta-w-1}}{(\beta-j)^\beta} dx \mathcal{H}_G(w; \beta),$$

we have

$$\sum_{w=0}^{\beta-1} \binom{\beta-1}{w} \beta \frac{(\frac{1}{\beta-j})^{-w} (\beta-j)^{\beta-w} \Gamma(\beta-w) \Gamma(w+1)}{(\beta-j)^\beta} \frac{{}_1F1[w+1, \beta+1, (\beta-j)it]}{\Gamma(\beta+1)} \mathcal{H}_G(w; \beta).$$

We now carry out the binomial coefficient formula and express it as discrete gamma function notation

$$\binom{\beta-1}{w} \beta = \frac{(\beta-1)! \beta}{(\beta-w-1)! w!} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-w) \Gamma(w+1)}.$$

Subsequently, substitute this last outcome into the penultimate expression and simplify to complete the proof of Equation (27). Proving (28) is simpler. Follow the proof of Proposition 2. $\square$

Remark 1. To find the mixed joint characteristic function

$$\varphi_{XW}(t, s) = E[\exp(i(tX + sW))],$$

where i is the imaginary unit, is a challenge exercise.

From these characteristic functions, one can derive all traditional concepts of probability distributions. Namely, all the moments of marginals of $\mathcal{H}_G(x, w; \beta, j)$ and joint moments can be generated from $\varphi(\cdot)$. We leave this challenge to the reader.

We may keep computing many more consequences and properties³ to describe the $\mathcal{H}_G(x, w; \beta, j)$ mixed joint probability distribution, but these are enough for our purposes in this paper and the following section. We have purposefully left some algebraic operations for the reader to perform in order to enhance their comprehension of the $\mathcal{H}$ distributions and to help them identify additional properties and applications across a variety of scientific disciplines. This is an important point to emphasize.

4. Example of decimal system $\beta = 10$

In this section, our purpose is to clarify the properties and usefulness of $\mathcal{H}_G(X, W; 10, j)$. We summarize the results in Tab. 1. From (16) for $\beta = 10$, summing over w , we have

$$\mathcal{H}_G(X; 10, j) = \sum_{w=0}^9 \frac{\binom{9}{w} 10x^w (10-j-x)^{9-w}}{(10-j)^{10}} \psi_{10}(w), \quad j = 0, 1, 2, \dots, 9. \quad 0 \leq x \leq 10-j.$$

Is is clear that there are 10 probability distributions; one homogeneous polynomial of degree 9 for each j in $\{0, 1, 2, \dots, 9\}$. See Tab. 1. To get the $E(X; 10, j)$ and $Var(X; 10, j)$ of $\mathcal{H}_G(X; 10, j)$ let's write Equations (22) and (23) as follows

$$(i) \quad E(X; \beta, j) = (\beta-j) \Delta_1(\beta) \quad \text{and} \quad (ii) \quad Var(X; \beta, j) = (\beta-j)^2 \Delta_2(\beta) - [(\beta-j) \Delta_1(\beta)]^2 \quad (29)$$

where

$$\Delta_1(\beta) = \frac{1}{(\beta+1) \log(\beta+1)} \log\left[\frac{(\beta+1)^\beta}{\beta!}\right]$$

and

$$\Delta_2(\beta) = \frac{1}{(\beta+2)(\beta+1) \log(\beta+1)} \log\left[\frac{(\beta+1)^{\beta(\beta+1)}}{\prod_{i=1}^{\beta} i^{2i}}\right].$$

Thus for $\beta = 10$ we have

$$\Delta_1(10) = \frac{1}{11 \log(11)} \log\left[\frac{11^{10}}{10!}\right] \approx 0.3364518821$$

and

$$\Delta_2(10) = \frac{1}{132 \log(11)} \log\left[\frac{11^{110}}{1^2 2^4 3^6 4^8 \dots 10^{20}}\right] \approx 0.1883053492.$$

Replacing $\Delta_1(10)$ and $\Delta_2(10)$, in (29), we get 10 different expected values, 10 variances, 10 covariances, one for each j and a single correlation coefficient, due to (40). See Tab. 1. Simplifying (i) and (ii) of (29) with (Maple.Soft, 2024) we get simpler formulas

$$E(x; \beta, j) = (\beta-j) \Delta_1(\beta) = 3.364518821 - 0.3364518821j, \quad j = 0, 1, \dots, 9,$$

³We trust that readers will have the opportunity to continue developing more properties and applications of this machine built for the advancement of probability and statistics and related fields.

$$Var(x; \beta, j) = (\beta - j)^2 [\Delta_2(\beta) - \Delta_1^2(\beta)] = 0.07510548034(-10 + j)^2, \quad j = 0, 1, 2, \dots, 9.$$

Now we describe the relationships between X and W . From Equations (25) and (26) we have

$$Cov(X, W; 10, j) = \frac{(10-j) \log \left[\frac{(11)^{(9)(10)}}{\prod_{i=1}^{10} i^{(2i-2)}} \right]}{(11) \log(11)} - \frac{(10-j) \log \left[\frac{(11)^{10}}{10!} \right] \log \left[\frac{(11)^{(9)}}{10!} \right]}{(11)(\log(11))^2}, \text{ for } j = 0, \dots, 9.$$

Evaluating at each j in $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and rounding we get:

j	0	1	2	3	4	5	6	7	8	9
$Cov(\cdot)$	6.780	6.102	5.424	4.746	4.068	3.390	2.712	2.034	1.356	0.678

Another way to find these results is by simplifying with (Maple.Soft, 2024) the $Cov(\cdot)$ for j and get

$$Cov(X, W; 10, j) = 6.780137493 - 0.6780137493 \times j \quad \text{for } j = 0, 1, 2, \dots, 9,$$

from which one can verify these results.

Next we calculate the correlation coefficient with Equation (26), for $j = 0, 1, \dots, 9$:

$$Corr(X, W; 10, j) = \frac{Cov(X, W; 10, j)}{\sqrt{\left[\frac{(10-j)^2 \log \left[\frac{(11)^{(10)(11)}}{\prod_{i=1}^{10} i^{(2i-1)}} \right]}{(11)(12) \log(11)} - \frac{(10-j)^2 (\log \left[\frac{(11)^{10}}{10!} \right])^2}{(11)^2 (\log(11))^2} \right] \left[\frac{\log \left[\frac{(11)^{(9)(9)}}{\prod_{i=1}^{10} i^{(2i-3)}} \right]}{\log(11)} - \frac{(\log \left[\frac{(11)^{(9)}}{10!} \right])^2}{(\log(11))^2} \right]}}.$$

Simplifying, with Maple⁴, surprisingly we found that $Corr(X, W; 10, j) = 0.9059142578 \text{ csgn}(10 - j)$ or

$$Corr(X, W; 10, j) = \frac{1.336129624(6.780137493 - 0.6780137493j)}{\sqrt{(10-j)^2}} \quad \text{for } j = 0, 1, 2, \dots, 9.$$

We note that all $Cov(X, W; 10, j)$ are positive and decay linearly as j gets closer to its highest value, $j \rightarrow \beta - 1$, and $Corr(X, W; 10, j) = +0.90591425$ are also positive invariant for all j in $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ since $\text{csgn}(z)$ is ± 1 and all covariances are positive. We explain this fact above and further in the lines below of Equation (26).

Additionally, with the Proposition 7, Equation (27), we get the characteristic function⁵

$$\varphi_X(t; \beta, j) = \sum_{w=0}^9 1F1[w+1, \beta+1, (\beta-j)it] \mathcal{H}_G(W; \beta) \quad \text{for each } j = 0, 1, 2, \dots, 9,$$

where i is the imaginary unit of complex numbers. In particular, if $\beta = 10$, we have

$$\varphi_X(t; 10, j) = \sum_{w=0}^9 1F1[w+1, 11, (10-j)it] \mathcal{H}_G(W; 10) \quad \text{for each } j = 0, 1, 2, \dots, 9.$$

Table 1

Ten probability distributions $\mathcal{H}_G(X; 10, j)$ (left column), one for each j , and some properties for $\beta = 10$

j	$\mathcal{H}_G(X; 10, j)$	$E(X; 10)$	$Var(X; 10)$
0	$\sum_{w=0}^9 \binom{9}{w} \frac{10x^w (10-x)^{9-w}}{10^{10}} \psi_{10}(w), 0 \leq x \leq 10$	$10\Delta_1(10)$	$100\Delta_2(10) - [10\Delta_1(10)]^2$
1	$\sum_{w=0}^9 \binom{9}{w} \frac{10x^w (9-x)^{9-w}}{9^{10}} \psi_{10}(w), 0 \leq x \leq 9$	$9\Delta_1(10)$	$81\Delta_2(10) - [9\Delta_1(10)]^2$
2	$\sum_{w=0}^9 \binom{9}{w} \frac{10x^w (8-x)^{9-w}}{8^{10}} \psi_{10}(w), 0 \leq x \leq 8$	$8\Delta_1(10)$	$64\Delta_2(10) - [8\Delta_1(10)]^2$
3	$\sum_{w=0}^9 \binom{9}{w} \frac{10x^w (7-x)^{9-w}}{7^{10}} \psi_{10}(w), 0 \leq x \leq 7$	$7\Delta_1(10)$	$49\Delta_2(10) - [7\Delta_1(10)]^2$
4	$\sum_{w=0}^9 \binom{9}{w} \frac{10x^w (6-x)^{9-w}}{6^{10}} \psi_{10}(w), 0 \leq x \leq 6$	$6\Delta_1(10)$	$36\Delta_2(10) - [6\Delta_1(10)]^2$
5	$\sum_{w=0}^9 \binom{9}{w} \frac{10x^w (5-x)^{9-w}}{5^{10}} \psi_{10}(w), 0 \leq x \leq 5$	$5\Delta_1(10)$	$25\Delta_2(10) - [5\Delta_1(10)]^2$
6	$\sum_{w=0}^9 \binom{9}{w} \frac{10x^w (4-x)^{9-w}}{4^{10}} \psi_{10}(w), 0 \leq x \leq 4$	$4\Delta_1(10)$	$16\Delta_2(10) - [4\Delta_1(10)]^2$
7	$\sum_{w=0}^9 \binom{9}{w} \frac{10x^w (3-x)^{9-w}}{3^{10}} \psi_{10}(w), 0 \leq x \leq 3$	$3\Delta_1(10)$	$9\Delta_2(10) - [3\Delta_1(10)]^2$
8	$\sum_{w=0}^9 \binom{9}{w} \frac{10x^w (2-x)^{9-w}}{2^{10}} \psi_{10}(w), 0 \leq x \leq 2$	$2\Delta_1(10)$	$4\Delta_2(10) - [2\Delta_1(10)]^2$
9	$\sum_{w=0}^9 \binom{9}{w} \frac{10x^w (1-x)^{9-w}}{1^{10}} \psi_{10}(w), 0 \leq x \leq 1$	$1\Delta_1(10)$	$1\Delta_2(10) - [1\Delta_1(10)]^2$

Based on the results of distinct cases ($\beta = 10$ is included), we dare to **Conjecture**: Let $Cov(X, W; \beta, j) = a \pm bj$ for any base $\beta \geq 2$ and let a and b be real numbers, $j = 0, 1, 2, \dots, \beta - 1$, then $Corr(X, W; \beta, j) = r \cdot \text{csgn}(\beta - j)$, where $|r| \leq 1$. The proof or disproof is in progress.

⁴The $\text{csgn}(z)$ function is used to determine in which half-plane ('left' or 'right') the complex-valued expression or number z lies.

⁵We leave the study and analysis of the characteristic function $\varphi_{XW}(t, s; \beta, j)$ of $\mathcal{H}_G(X, W; \beta, j)$ to the interested reader, which we will develop in another project.

We also illustrate some properties of the marginal distribution $\mathcal{H}_G(W; \beta, j)$ for $\beta = 10$. By Proposition 1 and Equations (6) and (7) we have

$$E(W; 10) = \frac{1}{\log(11)} \log\left[\frac{11^9}{10!}\right] \approx 2.70097070, \quad (30)$$

$$\text{Var}(W; 10) = \frac{1}{\log(11)} \log\left[\frac{11^{81}}{2^1 3^3 4^5 5^7 \dots 10^{17}}\right] - \left(\frac{1}{\log(11)} \log\left[\frac{11^{81}}{10!}\right]\right)^2 \approx 0.08729569, \quad (31)$$

and using Equation (28) from Proposition 7, we get

$$M_W(t) = \frac{1}{\log(11)} \log\left[\frac{11^{e^{9it}}}{2^{e^{it}-1} 3^{e^{2it}-e^{it}} \dots 10^{e^{9it}-e^{8it}}}\right]. \quad (32)$$

We observe that the behavior and properties of the marginal distribution $\mathcal{H}_G(W; \beta, j)$ are independent of j and are solely dependent on the parameter β . Consequently, they are distinct for each number system. In Tab. 1, each row constitutes a probability distribution over its specific sample space, pile nested lattices, $\{w\} \times [10 - j]$, for $w = 0, 1, 2, \dots, 9$ and each j in $\{0, 1, 2, \dots, 9\}$. Each p.d. is a homogeneous polynomial of degree 9, with 10 monomial terms⁶ and they satisfy the universal properties. Each term has an initial original contribution to the probability distribution. For instance, for $j = 0$, $0 \leq x \leq 10$ expanding the probability distribution $\mathcal{H}_G(x, w; 10, 0)$ given in the Tab. 1, then integrating each term, we have

$$\begin{aligned} \mathcal{H}_G(x, w; 10, 0) &= \overbrace{\frac{\log(2)}{10^9 \log(11)} (10 - x)^9}^{w=0} + \overbrace{\frac{9 \log(3/2)}{10^9 \log(11)} x (10 - x)^8}^{w=1} + \dots + \overbrace{\frac{\log(11/10)}{10^9 \log(11)} x^9}^{w=9}, \\ \int_0^{10} \mathcal{H}_G(x, w; 10, 0) dx &= \frac{\ln(2)}{\ln(11)} + \frac{\ln(3/2)}{\ln(11)} + \frac{\ln(4/3)}{\ln(11)} + \dots + \frac{\ln(11/10)}{\ln(11)} = \\ &= 0.2890648263 + 0.1690920837 + 0.1199727426 + \dots + 0.03974743221 = 1 \end{aligned}$$

or $\int_0^{10} \mathcal{H}_G(x, w; 10, 0) dx = 29 \% + 17 \% + 12 \% + \dots + 4 \% = 100 \%$.

See Fig. 2, which is an illustration for $j = 0, 3, 5, 9$. One can verify that $E(x, 10, j)$, $\text{Var}(x; 10, j)$, $\text{Cov}(x; 10, j)$ decay linearly, as $j \rightarrow \beta$, from the left, and β large. Each figure in Fig. 2 shows 10 smooth curves, one for each term of a homogeneous polynomial of degree 9. Each curve represents the path of a polynomial term obtained by replacing each value of $w \in \{0, 1, 2, \dots, 9\}$ in Equation (16). See also Tab. 1.

The distribution $\mathcal{H}_G(x, w; \beta, j)$ or more easily $\mathcal{H}_G(w; \beta)$ gives us precisely the contribution of each value of the random variable W and surprisingly, those are the Benford distribution probabilities, but counting all digits (0 is included) of a decimal system. It means, in finance, each value of W has a role to allocate a proportion of resources or investments. This function is non-negative and satisfies the universal property in its domain $[0, 10]$ for $j = 0$. For others values of $j = 1, 2, \dots, 9$, the same probabilities are redistributed. That is, for example, for $j = 9$ the distribution $\mathcal{H}_G(x, w; 10, 9)$ is on $[0, 1] \times \{w\}$ and integrating⁷ we have

$$\begin{aligned} \mathcal{H}_G(x, w; 10, 9) &= 10 \overbrace{\frac{\log(2)}{\log(11)} (1 - x)^9}^{w=0} + \overbrace{\frac{90 \log(3/2)}{\log(11)} x (1 - x)^8}^{w=1} + \dots + \overbrace{\frac{10 \log(11/10)}{\log(11)} x^9}^{w=9}, \\ \int_0^1 \mathcal{H}_G(x, w; 10, 9) dx &= 0.2890648263 + 0.1690920837 + \dots + 0.03974743221 = 1 \end{aligned}$$

or $\int_0^1 \mathcal{H}_G(x, w; 10, 9) dx = 29 \% + 17 \% + 12 \% + \dots + 4 \% = 100 \%$. We can arrange these integration results in the rows and columns of a 10×10 matrix whose entries are transition probabilities of a Markov chain with $\beta = 10$ states. The family of distributions $\mathcal{H}_G(x, w; 10, j)$ have 100 monomial terms, each of degree 9, and 10 homogeneous polynomials⁸ or probability distributions in their own probability space $(\Omega_{10}, \mathcal{F}_{10}, \mathbb{P}_{10})$ given by the nested, closed, and bounded intervals $[0, 10] \supset [0, 9] \supset \dots \supset [0, 1]$. These intervals are topological spaces with a rich set of properties, see (Munkres, 2000). Because the mixed variables in our study, namely, X is continuous and W is discrete, the underlying probability spaces are filtrations of a series of nested lattices. For instance, if $j = 0$ there is a rectangular lattice, with the right-hand rule orientation, $[0, 10] \times \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ with vertices at $(0, 0)$, $(10, 0)$, $(10, 9)$ and $(0, 9)$ and if $j = 9$ there is a rectangular lattice $[0, 1] \times \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ with vertices at $(0, 0)$, $(1, 0)$, $(1, 9)$ and $(0, 9)$. In Fig. 2, we provide a 3D geometric representation of $\mathcal{H}_G(x, w; 10, j)$ for $j = 0, 3, 5, 9$ whose lattices have right side continuous frontier intervals $0 \leq x \leq 10$, $0 \leq x \leq 7$, $0 \leq x \leq 5$ and $0 \leq x \leq 1$, respectively.

⁶The curve described by each monomial is the path followed by an object with an initial and final state. For $\beta = 10$, there are ten simple paths, which simultaneously change over time according to the change in each value of j and W in the $\mathcal{H}_G(X, W; 10, j)$ distribution. See Fig. 2. We might think that these changes describe random motions over time. But polynomials are smooth functions and follow the rules of standard calculus,

⁷Also, the same allocations are obtained evaluating the marginal $\mathcal{H}_G(w; \beta = 10)$ for each $w = 0, 1, \dots, 9$.

⁸In this research, homogeneous polynomials are smooth functions in a single variable that have nonzero β terms, each of degree $\beta - 1$, and obey to all calculus laws.

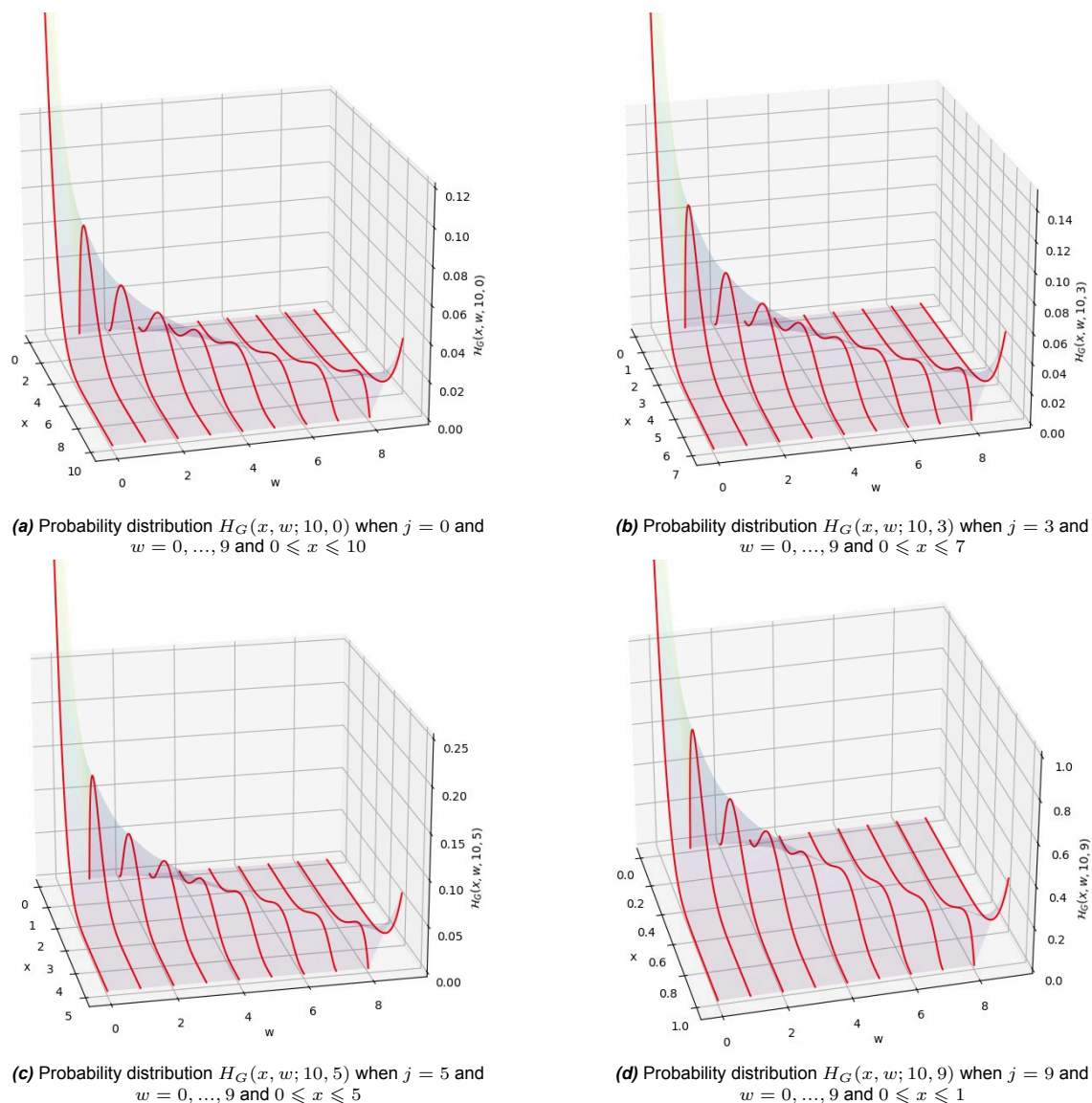


Fig. 2. The subfigures, (a)-(d), illustrates the shapes of four probability distributions for $j = 0, 3, 5, 9$, for a fixed $\beta = 10$

All these closed and bounded intervals are topological spaces in the Euclidean space, and satisfy several topological properties of invariance, such as compactness, connectedness, path-connectedness, locally connectedness, simple connectedness, contractibility, homotopy equivalence, and are simplicial complexes of dimension 1. Any interval $[0, \beta - j]$ for $j = 0, 1, 2, \dots, \beta - 1$ can be deformed continuously into $[0, 1]$. Simply, the variable $s = \frac{x}{\beta - j}$ maps x in $[0, \beta - j]$ to s in $[0, 1]$. Nicely, each space $[0, \beta - j]$ is contractible and behaves as a single point. Therefore, all have the same homotopy type, all are correlated, for any positive integers β and j . For our purpose, let $[0, \beta - j]$ be a contractible topological space with standard topology. Then all maps $\mathcal{H}_G : [0, \beta - j] \mapsto [0, \frac{\beta - j}{2}]$ are homotopic if and only if $[0, \frac{\beta - j}{2}]$ is contractible. In particular, β and j are both even or both odd. With this support, we may construct a family of homotopies as follows.

Let $[0, \beta - j]$ and $[0, \frac{\beta - j}{2}]$ be topological spaces, and let $\mathcal{H}_{G1}(x, w; \beta, j)$ and $\mathcal{H}_{G2}(x, w; \beta, j)$ be polynomials maps $\mathcal{H}_G : [0, \beta - j] \mapsto [0, \frac{\beta - j}{2}]$ of degree $\beta - 1$. Then a polynomial $\mathcal{H}_{G1}$ is homotopic to polynomial $\mathcal{H}_{G2}$ if there exists a continuous map $\mathbb{H} : [0, \beta - j] \times I \mapsto [0, \frac{\beta - j}{2}]$ such that $\mathbb{H}(x, 0) = \mathcal{H}_{G1}(x)$ and $\mathbb{H}(x, 1) = \mathcal{H}_{G2}(x)$ for all x in $[0, \beta - j]$. The map $\mathbb{H}$ is called homotopy from $\mathcal{H}_{G1}(x)$ to $\mathcal{H}_{G2}(x)$. For $(x, t) \in [0, \beta - j] \times I$, we may interpret $t \in I = [0, 1]$ as a parameter measuring time. Then $\mathbb{H}(x, t)$ is 1-parameter family of maps $[0, \beta - j] \mapsto [0, \frac{\beta - j}{2}]$. As time increases from 0 to 1 the map $\mathcal{H}_{G1}(x)$ is deformed continuously into the map $\mathcal{H}_{G2}(x)$.

In particular, for $\beta = 10, j = 0, 2, 4, 6, 8$ if we choose $j = 0$, then $\mathbb{H}(x, t) : [0, 10] \times I \mapsto [0, 5]$ and from Tab. 1 it corresponds to $\mathcal{H}(x, w; 10, 0) = \sum_{w=0}^9 \binom{9}{w} \frac{10x^w(10-x)^{9-w}}{10^{10}} \psi_{10}(w)$ and $\mathcal{H}(x, w; 10, 5) = \sum_{w=0}^9 \binom{9}{w} \frac{10x^w(5-x)^{9-w}}{5^{10}} \psi_{10}(w)$.

Then, a linear homotopy can be given by

$$\mathbb{H}(x, t) = (1 - t) \cdot \mathcal{H}_{G1}(x, w; 10, 0) + t \cdot \mathcal{H}_{G2}(x, w; 10, 5). \quad (33)$$

For every $x \in [0, 10]$, the line segment joining $\mathcal{H}_{G1}(x, w; 10, 0)$ lies completely inside $[0, 5]$. Essentially we move values

of the probability distribution $\mathcal{H}_{G1}(x, w; 10, 0)$ into probability distribution $\mathcal{H}_{G2}(x, w; 10, 5)$ along a straight line. In general, it is well known that homotopy is an equivalence relation. With regret, we leave this topological discussion of the sample space of these distributions, but those interested could read any textbook of algebraic topology. For example, (Munkres, 2000, Chap. 9). This specific example is merely one among hundreds or possibly thousands of other number systems that can be investigated. We hope to keep working on this topic and release its results shortly.

Discussion and conclusions

The outcomes of this research provide a new class of two-dimensional mixed probability distributions and their characteristics. The development of these distributions focused on both the general theoretical foundation and their practical implications in different scientific fields, particularly in statistics.

To achieve our findings: we first consider the F. Benford distribution in (1) and extend its domain from $\{1, 2, \dots, \beta - 1\}$ to $\{0, 1, 2, \dots, \beta - 1\}$ as shown in (5) and then embedding these generalized Benford distributions into an extended form of Bernstein polynomials. It is mathematically demonstrated that these models satisfy all the universal properties that every probability distribution must fulfill.

A key feature of these distributions is that all moments are functions only of their parameter $\beta \geq 2$, which is the base of a number system. Since there are infinitely many number systems, we are faced with an infinite family of probability distributions, all well-defined and supported mathematically. Here, we only present the finite family's case. The cornerstone of the development of these families of distributions lies in the Theorems 2 and (3), whose proofs and the implicated truth propositions guarantee their theoretical rigor. Our study is rigorously documented and is easy to understand for an advanced undergraduate audience in probability theory.

This "machine", known as the $\mathcal{H}$ distribution, is advantageous for the production of statistical congruence data for arbitrary congruence modules under standard arithmetic operations of addition and/or multiplication. The result presented in the context of the decimal system scenario with $\beta = 10$ indicates that we have developed a method for generating doubly stochastic Markov chains. Once the statistical data is made and put into the Markov chain's transition matrix, it can be used to study how a stochastic system changes over time and make predictions, among other things.

We refrain from employing historical data and acknowledge that our theoretical probability distributions require more tangible real-world applications, as they are novel and in the exploratory phase. Based on the results, we modestly believe that these distributions have a significant application impact across several scientific domains.

The main contribution of this study is to give an analytical tool that is easy to use and understand, capable of solving many problems in different fields, especially in statistics. Based on the mix of variables, the type of applications that could be focused on are amphoteric phenomena. That is, experiments or phenomena that behave continuously and discretely at the same time or alternately. This is similar to what occurs in price valuation in finance, with continuous models with discrete jumps. During the writing of the research, the following contributions were observed and noted: first. We have extended the Benford distribution to more general domains by including the digit zero in its definition for any number system with base $\beta \geq 2$. Second. The marginal distribution $\mathcal{H}_G(x; \beta, j)$ in (20) is related to the family of Bézier curves of degree $\beta - 1$ with β control points given by $\psi_\beta(w)$. Third. A new family of probability distributions $\mathcal{H}_G(X, W; \beta, j)$ is introduced into the statistical literature. Fourth. We introduce a methodology using these models to simulate statistical congruence data, specifically focusing on the operations of addition and multiplication within congruence classes across various number systems. Fifth. We provide a mathematical allocation function, which can be represented by either the $\mathcal{H}_G(W, \beta)$ or $\mathcal{H}_G(X, W; \beta, j)$ distribution, to optimize resource distribution in statistical models.

Also, we have demonstrated that all moments (expected value, variance, covariance, correlation coefficient, etc.) of the $\mathcal{H}$ distributions depend only of the parameter β . Therefore, they are easy to calculate. Future research should focus on refining this family of models with real-world data and finding new applications.

Finally, there are other applications of the $\mathcal{H}$ distributions that were planned for inclusion in this project, but there is a limitation of space in the document. Therefore, we are aware that we are leaving several parts of our study open, as they were not included in our objectives, and we believe this will be a source of inspiration for readers who decide to investigate further.

APPENDIX: Proofs

Proof of Proposition 6. We recall the well-known expressions of covariance and the cross-correlation coefficient. Recall that for $\beta \geq 2, j, w = 0, 1, \dots, \beta - 1$:

$$\sigma_{XW} = Cov(X, W; \beta, j) = E(XW) - E(X)E(W), \quad (34)$$

$$\rho_{XW} = Corr(X, W; \beta, j) = \frac{Cov(X, W; \beta, j)}{\sqrt{[Var(X; \beta, j)][Var(W; \beta, j)]}}. \quad (35)$$

We use (Maple.Soft, 2024) to solve some summations and integrations, as needed. Results of $E(X) = E(x; \beta, j)$ and $E(W) = E(w; \beta, j)$ are given in Equation (22) and Equation (6), respectively. Next, we compute $E(XW)$:

$$E(XW) = E(XW, \beta, j) = \frac{(b-j) \log \left[\frac{(\beta+1)^{(\beta-1)\beta}}{\prod_{i=1}^{\beta} i^{2i-2}} \right]}{(\beta+1) \log(\beta+1)}. \quad (36)$$

Thus, replacing Equations (6), (22) and (36) into (34) we have the covariance given in Equation (25), which is

$$Cov(X, W; \beta, j) = \frac{(\beta - j) \log \left[\frac{(\beta+1)^{(\beta-1)\beta}}{\prod_{i=1}^{\beta} i^{(2i-2)}} \right]}{(\beta+1) \log(\beta+1)} - \frac{(\beta - j) \log \left[\frac{(\beta+1)^{\beta}}{\beta!} \right] \log \left[\frac{(\beta+1)^{(\beta-1)}}{\beta!} \right]}{(\beta+1) (\log(\beta+1))^2}, j = 0, 1, \dots, \beta - 1. \quad (37)$$

On the other hand, we have computed the variances $Var(X; \beta, j)$ in Equation (23) and $Var(W; \beta, j)$ in Equation (7).

Thus, we replace Equations (7), (23) and (37) into (35) as follows

$$\begin{aligned} Corr(X, W; \beta, j) &= \\ &= \frac{\frac{(\beta-j) \log \left[\frac{(\beta+1)^{(\beta-1)\beta}}{\prod_{i=1}^{\beta} i^{(2i-2)}} \right]}{(\beta+1) \log(\beta+1)} - \frac{(\beta-j) \log \left[\frac{(\beta+1)^{\beta}}{\beta!} \right] \log \left[\frac{(\beta+1)^{(\beta-1)}}{\beta!} \right]}{(\beta+1) (\log(\beta+1))^2}}{\sqrt{\left[\frac{(\beta-j)^2 \log \left[\frac{(\beta+1)^{\beta(\beta+1)}}{\prod_{i=1}^{\beta} i^{2i}} \right]}{(\beta+1)(\beta+2) \log(\beta+1)} - \frac{(\beta-j)^2 (\log \left[\frac{(\beta+1)^{\beta}}{\beta!} \right])^2}{(\beta+1)^2 (\log(\beta+1))^2} \right] \left[\frac{\log \left[\frac{(\beta+1)^{(\beta-1)^2}}{\prod_{i=1}^{\beta} i^{(2i-3)}} \right]}{\log(\beta+1)} - \frac{(\log \left[\frac{(\beta+1)^{(\beta-1)}}{\beta!} \right])^2}{(\log(\beta+1))^2} \right]}}, j = 0, \dots, \beta - 1. \end{aligned} \quad (38)$$

Simplifying the common factors, $(\beta - j)$, of the numerator and denominator, we have

$$\begin{aligned} Corr(X, W; \beta, j) &= \\ &= \frac{\frac{\log \left[\frac{(\beta+1)^{(\beta-1)\beta}}{\prod_{i=1}^{\beta} i^{(2i-2)}} \right]}{(\beta+1) \log(\beta+1)} - \frac{\log \left[\frac{(\beta+1)^{\beta}}{\beta!} \right] \log \left[\frac{(\beta+1)^{(\beta-1)}}{\beta!} \right]}{(\beta+1) (\log(\beta+1))^2}}{\sqrt{\left[\frac{\log \left[\frac{(\beta+1)^{\beta(\beta+1)}}{\prod_{i=1}^{\beta} i^{2i}} \right]}{(\beta+1)(\beta+2) \log(\beta+1)} - \frac{(\log \left[\frac{(\beta+1)^{\beta}}{\beta!} \right])^2}{(\beta+1)^2 (\log(\beta+1))^2} \right] \left[\frac{\log \left[\frac{(\beta+1)^{(\beta-1)^2}}{\prod_{i=1}^{\beta} i^{(2i-3)}} \right]}{\log(\beta+1)} - \frac{(\log \left[\frac{(\beta+1)^{(\beta-1)}}{\beta!} \right])^2}{(\log(\beta+1))^2} \right]}}, j = 0, \dots, \beta - 1. \end{aligned} \quad (39)$$

It is clear that the correlation coefficient does not depend on j ; it only depends on the parameter $\beta \geq 2$. That is,

$$Corr(X, W, \beta) = \frac{\frac{\log \left[\frac{(\beta+1)^{(\beta-1)\beta}}{\prod_{i=1}^{\beta} i^{(2i-2)}} \right]}{(\beta+1) \log(\beta+1)} - \frac{\log \left[\frac{(\beta+1)^{\beta}}{\beta!} \right] \log \left[\frac{(\beta+1)^{(\beta-1)}}{\beta!} \right]}{(\beta+1) (\log(\beta+1))^2}}{\sqrt{\left[\frac{\log \left[\frac{(\beta+1)^{\beta(\beta+1)}}{\prod_{i=1}^{\beta} i^{2i}} \right]}{(\beta+1)(\beta+2) \log(\beta+1)} - \frac{(\log \left[\frac{(\beta+1)^{\beta}}{\beta!} \right])^2}{(\beta+1)^2 (\log(\beta+1))^2} \right] \left[\frac{\log \left[\frac{(\beta+1)^{(\beta-1)^2}}{\prod_{i=1}^{\beta} i^{(2i-3)}} \right]}{\log(\beta+1)} - \frac{(\log \left[\frac{(\beta+1)^{(\beta-1)}}{\beta!} \right])^2}{(\log(\beta+1))^2} \right]}}. \quad (40)$$

□

One can easily evaluate these formulas using Maple or any other software for any value of $\beta \geq 2$, as shown in the previous sections above.

Sources of funding. This study did not receive any grant from a funding institution in the public, commercial, or non-commercial sectors.

References

- Barabesi, L., & Pratelli, L. (2020). On the generalized Benford law. *Statistics Probability Letters*, 160, 108702. <https://doi.org/10.1016/j.spl.2020.108702>
- Benford, F. (1938). The law of anomalous numbers. *Proceedings of the American Philosophical Society*, 78(4), 551–572. <http://www.jstor.org/stable/984802>
- Debnath, L., & Basu, K. (2015). A short history of probability theory and its applications. *International Journal of Mathematical Education in Science and Technology*, 46(1), 13–39. <https://doi.org/10.1080/0020739X.2014.936975>
- Franklin, J. (2015). *The science of conjecture: evidence and probability before Pascal*. Baltimore, USA: Johns Hopkins University Press. <https://books.google.com.pe/books?id=1zDECQAQBAJ&lpg=PP1&hl=es&pg=PP1#v=onepage&q&f=false>
- Fu, D., Shi, Y. Q., & Su, W. (2007). A generalized Benford's law for JPEG coefficients and its applications in image forensics. In Delp III & Wong (Eds.), *Security, steganography, and watermarking of multimedia contents IX* (Vol. 6505, p. 65051L). SPIE. <https://doi.org/10.1117/12.704723>
- Maple.Soft. (2024). *Online Help (Calculus, sec. 5.4, page 133)*. <https://www.maplesoft.com/support/help/Maple/view.aspx?path=csn>.
- Mathematica.Soft. (2024). *Wolfram Language and System*. <https://reference.wolfram.com/language/ref/Integrate.html>.
- Munkres, J. (2000). *Topology* (2nd ed.). Upper Saddle River, USA: Prentice Hall.
- Newcomb, S. (1881). Note on the frequency of use of the different digits in natural numbers. *American Journal of Mathematics*, 4(1), 39–40. <https://doi.org/10.2307/2369148>
- Pietronero, L., Tosatti, E., Tosatti, V., & Vespignani, A. (2001). Explaining the uneven distribution of numbers in nature: the laws of Benford and Zipf. *Physica A: Statistical Mechanics and its Applications*, 293(1), 297–304. [https://doi.org/10.1016/S0378-4371\(00\)00633-6](https://doi.org/10.1016/S0378-4371(00)00633-6)
- Singer, I., & Thorpe, J. (1967). *Lecture notes on elementary topology and geometry*. New York, NY: Springer. <https://doi.org/10.1007/978-1-4615-7347-0>
- Tate, R. F. (1954). Correlation between a discrete and a continuous variable. point-biserial correlation. *The Annals of Mathematical Statistics*, 25(3), 603–607. <http://www.jstor.org/stable/2236844>
- Wiley.StatsRef. (2014). Point multiseriial (pms) correlation. In Balakrishnan, Colton, Everitt, Piegorsch, Ruggeri, & Teugels (Eds.), *Wiley statsref: Statistics reference online*. John Wiley Sons, Ltd. <https://doi.org/10.1002/9781118445112.stat00688>
- Wolfram.Soft. (2025). *Mathematica. Wolfram language & system documentation center*. <https://reference.wolfram.com/language/ref/Hypergeometric1F1.html>.

Отримано редакцію журналу / Received: 22.02.26
Прорецензовано / Revised: 14.04.26
Схвалено до друку / Accepted: 16.04.26
Опубліковано / Published: 05.06.26

Ксав'єр М. УАРКА, д-р філософії, проф.
ORCID ID: 0000-0002-1564-2343
e-mail: jhuarcao@usmp.pe
Університет Сан-Мартін-де-Поррес, Ліма, Перу

ЙМОВІРНІСНІ РОЗПОДІЛИ ТИПУ Н

Ймовірнісні розподіли є важливими моделями в усіх наукових галузях, однак вони мають вирішальне значення саме в статистиці. Ми пропонуємо новий клас змішаних сумісних розподілів ймовірностей, ймовірнісні простори яких є послідовностями вкладених $2D$ -решіток. Методологія побудови цих моделей передбачає вкладення дискретного розширеного розподілу Бенфорда з параметром β у неперервний модифікований поліном Бернштейна. β є параметром цього сімейства розподілів і визначає основу довільної системи числення. Метою роботи було описати математичну структуру та статистичні властивості взаємодій цих ймовірнісних розподілів. На основі теоретичних розробок та їхніх результатів ми робимо висновок, що зазначені моделі є цінними для численних застосувань у багатьох наукових дисциплінах. Для підтвердження гіпотези наведено конкретний приклад десяткової системи числення.

Ключові слова: розширений β -розподіл Бенфорда, поліноми Бернштейна, система чисел, універсальні властивості ймовірнісного розподілу, гомотопічний семплічний комплекс.

Автор заявляє про відсутність конфлікту інтересів. Спонсори не брали участі в розробленні дослідження; у зборі, аналізі чи інтерпретації даних; у написанні рукопису; в рішенні про публікацію результатів.

The author declares no conflicts of interest. The funders had no role in the design of the study; in the collection, analyses or interpretation of data; in the writing of the manuscript; in the decision to publish the results.