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Dedicated to the memory of academician Mykola Perestyuk

STABILITY OF INFECTION-FREE ALMOST PERIODIC SOLUTION IN IMPULSIVE SEIR-MODEL

Mathematical models of the infection spread dynamics have remained a relevant area of research for the past few decades. Moreover, these models have also proven to be suitable for modeling DDoS attacks in computer networks. The qualitative behaviour of solutions in such models is usually arranged as follows: there is some threshold value at which the global asymptotic stability of the solution with a zero component, which is responsible for the number of infected individuals, changes to the stability of the non-zero equilibrium position. From a meaningful point of view, this means that the self-exhaustion and disappearance of the infection changes to a constant circulation of the infection in the population.

However, the presence of an impulsive perturbation can make the infection-free solution asymptotically stable regardless of the threshold value, and such impulsive perturbations allow a natural interpretation within the framework of the epidemiological model as the process of vaccination of a certain part of individuals.

Such problems have already been considered for the SIR- and SEIR-models in the case of a simple structure of impulsive points, due to which the impulsive problem was actually reduced to a problem without impulsive perturbation. However, a more complex structure of the set and magnitudes of impulsive effects requires the application of the theory of impulsive differential equations, which is done in this work.

Namely, the article considers the SEIR-system of differential equations with impulsive perturbations that have an almost periodic character. The existence and properties of an infection-free impulsive almost periodic solution are investigated, and its asymptotic stability is proven.

Key words: SEIR-model of differential equations, impulsive perturbation, infection-free solution, periodic solution, almost periodic solution.

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Introduction

The classical SIR-model is a three-dimensional system of differential equations that describes the infection spread dynamics in a population consisting of potentially Susceptible, Infectious and Recovered individuals. Various modifications of this system, as well as a qualitative analysis, are contained in the review (Hethcote, 1989). SEIR-model is a four-dimensional system that includes Exposed individuals of infection as part of the population (Liu, Hethcote, & Levin, 1987). This model can also be adapted to describe the dynamics of certain types of DDoS attacks in computer networks (Balarezo et al., 2022). A qualitative analysis of this type of model has been investigated in works (Li, & Muldowney, 1995; Li, & Muldowney, 1996). As in SIR-models, it is shown that if the contact number $\sigma \leq 1$, then there are physically justified invariant regions where the only globally asymptotically stable equilibrium position is $I = 0$ (infection-free). If $\sigma > 1$, then another equilibrium position with $I > 0$ becomes asymptotically stable.

It turns out that the presence of an impulsive perturbation can radically change this qualitative picture. The theory of differential equations systems with an impulsive perturbation was created in the works of prominent mathematicians A. M. Samoilenko and M. O. Perestyuk (Samoilenko, & Perestyuk, 1995). In models of the infection spread, the impulsive perturbation has a natural interpretation as a vaccination procedure of a certain part of potentially susceptible individuals. For the first time, such a problem was considered in (Shulgin, Stone, & Agur, 1998). In the simplest situation, when moments of an impulsive perturbation are equidistant, the asymptotic stability of an infection-free impulsive periodic solution was proven. The same impulsive problem for the SEIR-model was solved in (D'Onofrio, 2002). It should be noted that the simple structure of the impulsive moments ($t_n = nT$) actually allows to reduce such a problem to the study of a periodic problem without impulses, that makes it problematic to use the approach (Shulgin, Stone, & Agur, 1998), (D'Onofrio, 2002) to more complex impulsive problems (Feketa, Klinshov, & Lücken, 2021).

In this paper, we investigate the impulsive problem when at the moments of time $t_n = nT + \alpha_n$, $(\{\alpha_n\})$ – almost periodic sequence) trajectories of SEIR-model undergo impulsive perturbation which can be interpreted as a vaccination process. Using the general impulsive systems theory (Samoilenko, & Perestyuk, 1995), we prove the existence of an almost periodic impulsive infection-free solution and investigate its dependence on the parameter T . This allows us to establish the main result of the paper that for sufficiently small T this solution is asymptotically stable.

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1. Formulation of the problem

We consider a classical SEIR-model respect to the unknown functions $S(t)$ (Susceptible is the fraction of the population that can become infected), $E(t)$ (Exposed is the fraction of the population that is carried the infection), $I(t)$ (Infectious is the fraction of the population that is infected) and $R(t)$ (the fraction of the population that Recovered and has immune), for which equality

$$S(t) + E(t) + I(t) + R(t) = 1, \quad t \geq 0$$

and an ordinary differential equations system

$$\begin{cases} \dot{S} = -\lambda SI + \nu - \nu S + \delta R, \\ \dot{E} = \lambda SI - (\varepsilon + \nu)E, \\ \dot{I} = \varepsilon E - (\gamma + \nu)I, \\ \dot{R} = \gamma I - (\delta + \nu)R, \end{cases}$$

holds. Here $\lambda, \nu, \delta, \varepsilon, \gamma$ are positive parameters that determine the indicators of the spread of infection, namely:

$\lambda > 0$ is the natural frequency of contacts,

$\nu > 0$ is the proportion of deaths and newborns,

$\delta > 0$ is the proportion of recovered people who lose immunity,

$\varepsilon > 0$ is the proportion of infection carriers who get sick,

$\gamma > 0$ is the proportion of sick people who recover.

Also important for analysing the problem is the contact number

$$\sigma = \frac{\lambda \varepsilon}{(\varepsilon + \nu)(\gamma + \nu)}.$$

Substituting the expression $R = 1 - S - E - I$ into the system, we have the main object of study

$$\begin{cases} \dot{S} = -\lambda SI + \nu - \nu S + \delta(1 - S - E - I), \\ \dot{E} = \lambda SI - (\varepsilon + \nu)E, \\ \dot{I} = \varepsilon E - (\gamma + \nu)I. \end{cases} \quad (1)$$

Let's

$$D = \{(S, E, I) \in \mathbb{R}_+^3 \mid 0 \leq S + E + I \leq 1\}$$

then the following result is known.

Theorem 1 (Li, & Muldowney, 1996). *For any initial data $(S(0), E(0), I(0)) \in D$ there is only one corresponding solution of (1) on the interval $[0, +\infty)$ and for all $t \geq 0$ the inclusion $(S(t), E(t), I(t)) \in D$ takes place.*

If $\sigma \leq 1$, then where is the only one equilibrium point $P_0^ = \{S_0^* = 1, E_0^* = 0, I_0^* = 0\}$ (infection free) in D , which is globally asymptotically stable in D .*

If $\sigma > 1$, then P_0^ is an unstable equilibrium point and another equilibrium point $P_+^* = \{S_1^* = \frac{1}{\sigma}, E_1^* \in (0, 1), I_1^* \in (0, 1)\}$ appears, which is globally asymptotically stable in $int D$.*

Modelling the vaccination process as a sharp decrease in the susceptible to infection population proportion we arrive at an impulsive problem

$$\Delta S|_{t=t_n} := S(t_n + 0) - S(t_n) = -b_n S(t_n), \quad (2)$$

where $\{b_n\} \subset (0, 1)$, $\{t_n\}$ are the given values and the moments of the impulsive perturbation.

By solution (1), (2) according to (Samoilenko, & Perestyuk, 1995) we will understand a left-continuous function, which is satisfies the system (1), when $t \neq t_n$, and at points $\{t_n\}$ satisfies the equality (2).

In papers (Shulgin, Stone, & Agur, 1998), (D'Onofrio, 2002) in the case $b_n = b$, $t_n = nT$ it is shown that when $\sigma > 1$ in problem (1), (2) there is a periodic impulsive solution $\bar{P}(t) = (\bar{S}(t), 0, 0)$, which under condition

$$\frac{1}{T} \int_0^T \bar{S}(t) dt < \frac{1}{\sigma}$$

is asymptotically stable.

In this paper we generalize this result to the case of almost periodic (Samoilenko, & Perestyuk, 1995) sequences $\{b_n\}$, $\{t_n = nT + \alpha_n\}$. We prove the existence of an almost periodic impulsive infection-free solution and show that for sufficiently small $T > 0$ this solution is asymptotically stable.

2. Main results

Let us first consider a periodic problem, namely, let there exist $p \in \mathbb{N}$ and $T > 0$ such that

$$\begin{aligned} 0 < t_1 < t_2 < \dots < t_p < T, \quad t_{n+p} = t_n + T, \quad n \geq 1, \\ b_{n+p} = b_n, \quad n \geq 1. \end{aligned} \quad (3)$$

Then the following statement is true.

Lemma 1. *Under conditions (3), problem (1), (2) has a periodic infection-free impulsive solution, and also*

$$\sup_{t \geq 0} \bar{S}(t) \rightarrow 0, \quad T \rightarrow 0. \quad (4)$$

Proof of Lemma 1. Let's find $\bar{S}(t)$ as a T -periodic solution to the problem

$$\begin{cases} \dot{S} = (\nu + \delta)(1 - S), \\ \Delta S|_{t=t_n} = -b_n S(t_n). \end{cases} \quad (5)$$

On the interval of continuity $[\tau, t]$ we have

$$S(t) = (S(\tau) - 1)e^{-(v+\delta)(t-\tau)} + 1,$$

then

$$S(t_1 + 0) = [(S(0) - 1)e^{-(v+\delta)t_1} + 1](1 - b_1) = S(0)e^{-(v+\delta)t_1}(1 - b_1) + \alpha_1, \quad 0 < \alpha_1 < 1 - e^{-(v+\delta)t_1},$$

and so on

$$S(t_p + 0) = S(0)e^{-(v+\delta)t_p} \prod_{i=1}^p (1 - b_i) + \alpha_p, \quad 0 < \alpha_p < 1 - e^{-(v+\delta)t_p}.$$

Therefore,

$$S(T) = S(0)e^{-(v+\delta)T} \prod_{i=1}^p (1 - b_i) + (\alpha_p - 1)e^{-(v+\delta)(T-t_p)} + 1$$

and the initial data for this periodic solution is determined as follows

$$S(0) = \frac{(\alpha_p - 1)e^{-(v+\delta)(T-t_p)} + 1}{1 - e^{-(v+\delta)T} \prod_{i=1}^p (1 - b_i)}. \quad (6)$$

At the same time, it follows from (6) that

$$S(0) = \frac{1 - e^{-(v+\delta)T}}{1 - e^{-(v+\delta)T} \prod_{i=1}^p (1 - b_i)},$$

which means that $S(T) \rightarrow 0, T \rightarrow 0$. Then from (5) we deduce that for any $\varepsilon > 0$ there exists $T^* > 0$ such that for all $T \in (0, T^*)$ and for all $t \geq 0$ we have $S(t) < \varepsilon$, hence (4) holds. Lemma 1 is proven. $\square$

Remark 1. It is easy to see that the periodic solution $\bar{S}(t)$ is asymptotically stable in problem (5), that is, for an arbitrary initial data $S(0) \in (0, 1)$ and the corresponding solution $S(t)$ of problem (5) holds

$$S(t) - \bar{S}(t) \rightarrow 0, \quad t \rightarrow \infty. \quad (7)$$

Now we consider an almost periodic impulsive problem. We will use the following definitions (Samoilenko & Perestyuk, 1995).

Definition 1. A sequence $\{\tau_i\}_{i \in \mathbb{Z}}$ is almost periodic if

$$\forall \varepsilon > 0 \quad \exists N = N(\varepsilon) \quad \forall k \in \mathbb{Z} \quad \exists p \in [k, k + N] \cap \mathbb{Z}: \quad \forall i \in \mathbb{Z} \quad |\tau_{i+p} - \tau_i| < \varepsilon. \quad (8)$$

The number p is called ε -almost period of the sequence $\{\tau_i\}_{i \in \mathbb{Z}}$.

In the sense of Definition 1, it is stated that for any $\varepsilon > 0$ there exists a relatively dense set of ε -almost periods.

Remark 2. If the sequence $\{t_i\}_{i \in \mathbb{Z}}$ is such that equality $t_{i+p} = t_i + T$ holds for any $i \in \mathbb{Z}$, then the sequence $\{\tau_i = t_{i+1} - t_i\}_{i \in \mathbb{Z}}$ is almost periodic.

In the following, we consider the situation when the sequence $\{t_i\}_{i \in \mathbb{Z}}$ satisfies the condition

$$\dots < t_i < t_{i+1} < \dots, \quad i \in \mathbb{Z}; \quad t_i \rightarrow \pm\infty, \quad i \rightarrow \pm\infty.$$

Let $N(t, t + s)$ denotes the number of the sequence $\{t_i\}_{i \in \mathbb{Z}}$ members belonging to the interval $(t, t + s]$.

It is known (Samoilenko, & Perestyuk, 1995) that if a sequence $\{t_{i+1} - t_i\}_{i \in \mathbb{Z}}$ is almost periodic, then uniformly over $t \in \mathbb{R}$ there exists a limit

$$\lim_{s \rightarrow \infty} \frac{N(t, t + s)}{s} = \theta. \quad (9)$$

In particular, if $t_{i+p} = t_i + T$ then $\theta = \frac{p}{T}$.

Definition 2. For a given sequence $\{t_i\}_{i \in \mathbb{Z}}$, we put $\tau_i^j = t_{i+j} - t_i, i, j \in \mathbb{Z}$. We will say that the sequence $\{\tau_i^j\}$ is uniformly almost periodic if for an arbitrary $\varepsilon > 0$ there is a relatively dense set of ε -almost periods common to all j , that is

$$\forall \varepsilon > 0 \quad \exists N \quad \forall k \in \mathbb{Z} \quad \exists p \in [k, k + N] \cap \mathbb{Z}: \quad \forall j \in \mathbb{Z} \quad \forall i \in \mathbb{Z} \quad |\tau_{i+p}^j - \tau_i^j| < \varepsilon.$$

Lemma 2 (Samoilenko, & Perestyuk, 1995). Let the sequence $\{t_i\}_{i \in \mathbb{Z}}$ be such that for some $T > 0$ the equalities

$$t_i = iT + \alpha_i, \quad i \in \mathbb{Z},$$

hold, where the sequence $\{\alpha_i\}_{i \in \mathbb{Z}}$ is almost periodic and $\sup_{i \in \mathbb{Z}} |\alpha_i| = \alpha < \frac{T}{2}$. Then the sequence $\{\tau_i^j = t_{i+j} - t_i\}$ is uniformly almost periodic.

Definition 3. A piecewise continuous function $\varphi: \mathbb{R} \rightarrow \mathbb{R}^d$ with discontinuities of the first kind at points $\{t_i\}_{i \in \mathbb{Z}}$ is called almost periodic if the sequence $\{\tau_i^j = t_{i+j} - t_i\}$ is uniformly almost periodic and the following properties are satisfied:

- 1) $\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall i \in \mathbb{Z} \quad \forall t', t'' \in (t_i, t_{i+1}): |t' - t''| < \delta \Rightarrow \|\varphi(t') - \varphi(t'')\| < \varepsilon;$
- 2) for arbitrary $\varepsilon > 0$ there exists set Γ for which there exists such $l > 0$ that for all $a \in \mathbb{R}: [a, a + l] \cap \Gamma \neq \emptyset$ and for each $\tau \in \Gamma, t \in \mathbb{R}$ inequality $|t - t_i| > \varepsilon$ leads to $\|\varphi(t + \tau) - \varphi(t)\| < \varepsilon$.

Now consider the almost periodic impulsive problem (2), where

$$t_i = iT + \alpha_i, \quad i \in \mathbb{Z},$$

$$\text{the sequence } \{\alpha_i\}_{i \in \mathbb{Z}} \text{ is almost periodic and } \sup_{i \in \mathbb{Z}} |\alpha_i| = \alpha < \frac{T}{2}, \quad (10)$$

$$\text{the sequence } \{b_i\}_{i \in \mathbb{Z}} \text{ is almost periodic, } \{b_i\}_{i \in \mathbb{Z}} \in (0, 1) \text{ and } \inf_{i \in \mathbb{Z}} |b_i| \geq b > 0.$$

Lemma 3. Under conditions (10), problem (1), (2) has an almost periodic infection-free solution $\bar{P}(t) = \{\bar{S}(t), 0, 0\}$ in the sense of Definition 3 and

$$\sup_{t \geq 0} \bar{S}(t) \rightarrow 0, \quad T \rightarrow 0.$$

Remark 3. It also follows from (Samoilenko, & Perestyuk, 1995, Theorem 24.1) that the almost periodic solution $\bar{S}(t)$ is asymptotically stable for problem (5)9, i.e., the limit equality (7) holds.

Remark 4. For the correctness of the definitions, we consider the impulsive perturbation on the entire real axis, but we are interested in the dynamics on the half-axis $[0, +\infty)$.

Proof of Lemma 3. Since the inequality

$$-(\nu + \delta) + \theta \ln \left(\sup_{i \in \mathbb{Z}} (1 - b_i) \right) \leq -(\nu + \delta) < 0$$

holds, where θ is determined from (8), then from (Samoilenko, & Perestyuk, 1995, Theorem 24.1) problem (5) has a unique almost periodic solution

$$\bar{S}(t) = (\nu + \delta) \int_{-\infty}^t X(t, s) ds, \quad (11)$$

where for $t_k \leq t < t_{k+1}$, $t_{m-1} \leq s < t_m$, $s < t$ we have

$$X(t, s) = e^{-(\nu + \delta)(t - t_k)} \prod_{s \leq t_i < t} (1 - b_i) e^{-(\nu + \delta)(t_i - t_{i-1})} (1 - b_m) e^{-(\nu + \delta)(t_m - s)}.$$

From formula (11) we deduce that $\bar{S}(t) \in (0, 1)$ for an arbitrary $t \in \mathbb{R}$.

For a more precise estimate of $\bar{S}(t)$, we note that from the existence of a uniform with respect to $t \in \mathbb{R}$ limits in (8) we have

$$\theta = \lim_{k \rightarrow \infty} \frac{N \left(0, kT + \frac{T}{2} \right)}{kT + \frac{T}{2}} = \lim_{k \rightarrow \infty} \frac{k + \min\{0, \text{sign } \alpha_0\}}{kT + \frac{T}{2}} = \frac{1}{T}.$$

Then for an arbitrary $\varepsilon > 0$ there exists such $K = K(\varepsilon) \geq 1$ that for all $t \geq s$ we have

$$X(t, s) \leq K e^{(\varepsilon - \nu - \delta + \frac{1}{T} \ln(1 - b))(t - s)},$$

and therefore (Samoilenko, & Perestyuk, 1995, Theorem 24.1) for $\varepsilon = \frac{\nu + \delta}{2}$ and for $K \geq 1$, which does not depend on T for sufficiently small T number $\beta = -\frac{\nu + \delta}{2} + \frac{1}{T} \ln(1 - b) < 0$ is negative and so for $t \geq 0$ we have

$$\bar{S}(t) \leq K(\nu + \delta) \cdot \left(\frac{1}{-\beta} \right) = \frac{K(\nu + \delta)}{\frac{\nu + \delta}{2} + \frac{1}{T} |\ln(1 - b)|},$$

therefore $\sup_{t \geq 0} \bar{S}(t) \rightarrow 0$, $T \rightarrow 0$. Lemma 3 is proven. $\square$

The main result of the paper is the following theorem.

Theorem 2. Let in problem (1), (2) impulsive perturbations satisfy conditions (3) or (10) and $\bar{P}(t) = \{\bar{S}(t), 0, 0\}$ is a corresponding infection-free impulsive almost periodic solution. Then there is such $T^* > 0$ that for $T < T^*$ the solution $\bar{P}(t)$ is globally (in $\text{Int } D$) asymptotically stable.

Proof of Theorem 2. With fixed $E(t)$ and $I(t)$, consider the impulsive problem for first equation of system (1).

$$\begin{cases} \dot{S} = -\lambda SI + (\nu + \delta)(1 - S) - \delta(E + I), \\ \Delta S|_{t=t_n} = -b_n S(t_n), \quad S|_{t=0} = S_0 \in (0, 1). \end{cases} \quad (12)$$

Since there are $E(t) \geq 0$, $I(t) \geq 0$, then for an arbitrary $t \geq 0$ we have

$$S(t) \leq X(t), \quad (13)$$

where $X(t)$ is the solution of (5) with the initial condition $X(0) = S_0$ and therefore, according to the remarks 1, 3, we have

$$X(t) - \bar{S}(t) \rightarrow 0, \quad t \rightarrow \infty. \quad (14)$$

Now, with a fixed $S(t)$, consider second and third equations of system (1)

$$\begin{cases} \dot{E} = \lambda SI - (\varepsilon + \nu)E, \\ \dot{I} = \varepsilon E - (\gamma + \nu)I, \\ E(0) = E_0 \in (0, 1), \quad I(0) = I_0 \in (0, 1). \end{cases} \quad (15)$$

Since system (15) is cooperative, that is, there are inequalities

$$\frac{\partial \dot{E}}{\partial I} = \lambda S(t) > 0, \quad \frac{\partial \dot{I}}{\partial E} = \varepsilon > 0,$$

hold, then according to Kamke's theorem, we have

$$0 \leq E(t) \leq Y(t), \quad 0 \leq I(t) \leq Z(t),$$

where $\{Y, Z\}$ are the solutions of the system

$$\begin{cases} \dot{Y} = \lambda XZ - (\varepsilon + \nu)Y, \\ \dot{Z} = \varepsilon Y - (\gamma + \nu)Z, \\ Y(0) = E_0, \quad Z(0) = I_0, \end{cases} \quad (16)$$

and $X(t)$ is satisfied by the limiting relation (14). So, for sufficiently large t , we have that

$$\frac{d}{dt} (Y(t) + Z(t)) = -\nu Y(t) - (\gamma + \nu - \lambda X(t))Z(t) \leq -\nu (Y(t) + Z(t)), \quad (17)$$

if T^* is chosen so that the inequality

$$\sup_{t \geq 0} \bar{S}(t) < \frac{\nu}{\lambda} \quad (18)$$

holds for $T < T^*$.

Thus, for arbitrary $t \geq 0$ we have

$$E(t) + I(t) \leq K(E_0 + I_0)e^{-\nu t}, \quad (19)$$

where $K > 0$ is some constant.

Let's go to the new variables x, y, z in (1), (2), where

$$\begin{aligned} S(t) &= \bar{S}(t) + x(t), \\ E(t) &= y(t), \\ I(t) &= z(t). \end{aligned}$$

Then the stability of $\bar{P}(t)$ is equal to the stability of the zero solution of the system

$$\begin{cases} \dot{x} = -\lambda \bar{S}(t)z - \lambda xz - \nu x + \delta(-x - y - z), \\ \dot{y} = \lambda \bar{S}(t)z + \lambda xz - (\varepsilon + \nu)y, \\ \dot{z} = \varepsilon y - (\gamma + \nu)z, \\ \Delta x|_{t=t_n} = -b_n x(t_n). \end{cases} \quad (20)$$

Under condition (18), for each $t \geq 0$ we have

$$y(t) + z(t) \leq (y_0 + z_0)e^{-\nu t}. \quad (21)$$

Further, on an arbitrary interval of continuity (τ, t) , from the first equation in (20) we obtain

$$\frac{1}{2} \frac{d}{dt} x^2(t) \leq -(\nu + \delta)x^2(t) - \delta x(t)y(t) - (\lambda \bar{S}(t) + \delta)x(t)z(t).$$

Choosing $T^* > 0$ so that for $T < T^*$ condition

$$\sup_{t \geq 0} \bar{S}(t) < \frac{\nu}{\lambda} \quad (22)$$

is fulfilled, we obtain

$$\frac{1}{2} \frac{d}{dt} x^2(t) \leq -(\nu + \delta)x^2(t) + (\nu + \delta)|x(t)|(y(t) + z(t)) \leq -\frac{\nu + \delta}{2}x^2(t) + \frac{\nu + \delta}{2}(y(t) + z(t))^2.$$

Thus, on an arbitrary interval of continuity (τ, t) , using (21), we derive

$$\begin{aligned} \frac{d}{dt} x^2(t) &\leq -(\nu + \delta)x^2(t) + (\nu + \delta)(y(t) + z(t))^2, \\ x^2(t) &\leq x^2(\tau)e^{-(\nu + \delta)(t - \tau)} + (\nu + \delta) \int_{\tau}^t (y_0 + z_0)^2 e^{-2\nu s} e^{-(\nu + \delta)(t - s)} ds \leq \\ &\leq x^2(\tau)e^{-(\nu + \delta)(t - \tau)} + \frac{\nu + \delta}{\delta} (y_0 + z_0)^2 e^{-\nu t}. \end{aligned} \quad (23)$$

Then

$$\begin{aligned} x^2(t_1) &\leq x_0^2 e^{-(\nu + \delta)t_1} + \frac{\nu + \delta}{\delta} (y_0 + z_0)^2 e^{-\nu t_1}, \\ x^2(t_2) &\leq x^2(t_1)(1 - b_1)^2 e^{-(\nu + \delta)(t_2 - t_1)} + \frac{\nu + \delta}{\delta} (y_0 + z_0)^2 e^{-\nu t_2} \leq \\ &\leq x_0^2 (1 - b_1)^2 e^{-(\nu + \delta)t_2} + \frac{\nu + \delta}{\delta} (y_0 + z_0)^2 ((1 - b_1)^2 + 1) e^{-\nu t_2}, \end{aligned}$$

continuing further at the k -th step, we obtain

$$x^2(t_k) \leq x_0^2 (1 - b)^{2k} e^{-(\nu + \delta)t_k} + \frac{\nu + \delta}{\delta(1 - b^2)} (y_0 + z_0)^2 e^{-\nu t_k}, \quad k \geq 1. \quad (24)$$

From estimates (24) we deduce that $x^2(t) \rightarrow 0, t \rightarrow \infty$, and this, together with (21), gives the desired asymptotic stability.

Theorem 2 is proven. $\square$

Discussion and conclusions

The article considers the SEIR system of differential equations with impulsive perturbations of an almost periodic nature. The existence and properties of an almost periodic impulsive solution without infections are investigated, and its asymptotic stability is also proven. From a practical point of view, this means that by choosing a certain realistic vaccination schedule (which takes into account possible deviations from a strictly periodic nature) and supporting vaccination of a certain proportion of the population, it is possible to achieve the stability of the infection-free state of the system, which will mean the attenuation of the epidemic process. The obtained results can also be applied to high-dimensional systems of differential equations, which are used for describing DDoS attacks in computer networks. The study generalizes previously known results by considering a more mathematically complex model that allows for a more realistic practical interpretation. It should be noted that the linear impulse perturbation condition (2) is due to empirical data on the cessation of the circulation of infections in the population upon reaching a sufficiently high level (80–90 %, and sometimes more than 95%) of vaccinated people in the population (see, for example, <https://ourworldindata.org/vaccination>). As a rule, achieving such levels requires mass vaccination, so we really neglect the throughput of the medical system, considering it proportional to the population size. In our further studies, we plan to consider the case of a nonlinear dependence of the magnitude of the impulse disturbance on S , which is a more realistic case from the point of view of the practical vaccination process, which depends on the capabilities of the system.

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ПРО СТІЙКІСТЬ МАЙЖЕ ПЕРІОДИЧНОГО ВІЛЬНОГО ВІД ІНФЕКЦІЇ РОЗВ'ЯЗКУ В ІМПУЛЬСНІЙ SEIR-МОДЕЛІ

Математичні моделі динаміки розповсюдження інфекції є актуальним напрямком досліджень упродовж кількох останніх десятиліть. Ба більше, ці моделі показали себе придатними також для моделювання DDoS-атак у комп'ютерних мережах. Якісна поведінка розв'язків у таких моделях зазвичай улаштована так: існує деяке порогове значення, у разі досягнення якого глобальна асимптотична стійкість розв'язку з нульовою компонентою, що відповідає за кількість інфікованих осіб, змінюється на стійкість ненульового положення рівноваги. Зі змістовної позиції це означає, що самостійне вичерпання та зникнення інфекції змінюється на постійну циркуляцію інфекції в популяції.

Однак наявність імпульсного збурення може зробити вільний від інфекції розв'язок асимптотично стійким незалежно від порогового значення, причому такі імпульсні збурення допускають природну інтерпретацію у межах епідеміологічної моделі як процес вакцинації певної частини індивідумів.

Такі задачі вчені вже розглядали для SIR- і SEIR-моделей у випадку простої структури імпульсних точок, завдяки чому імпульсна задача фактично зводилася до задачі без імпульсного збурення. Проте складніша структура множини та величин імпульсних впливів вимагає застосування теорії імпульсних диференціальних рівнянь, що і зроблено у цій роботі. Зокрема, у статті розглянуто SEIR-систему диференціальних рівнянь з імпульсними збуреннями, що мають майже періодичний характер. Досліджено існування та властивості вільного від інфекції імпульсного майже періодичного розв'язку та доведено його асимптотичну стійкість.

Ключові слова: SEIR-модель диференціальних рівнянь, імпульсне збурення, вільний від інфекції розв'язок, періодичний розв'язок, майже періодичний розв'язок.

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