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INCOMPLETE TRIGONOMETRIC SPLINES IN THE PROBLEMS OF CONSTRUCTING APPROXIMATE SOLUTIONS OF THE SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

This study introduces a numerical analytical method for deriving approximate solutions to the first boundary value problem associated with second-order linear differential equations. The approach is founded upon the utilization of incomplete fundamental trigonometric splines, both even and odd, whose structural properties closely parallel those of classical polynomial splines and have been extensively investigated in prior works. The proposed methodology employs residual minimization through the established collocation technique, thereby reducing the original problem to the resolution of systems of linear algebraic equations with respect to the spline-defining parameters. A distinctive feature of this method lies in the fact that the values of the sought solution themselves serve as the determining parameters of the fundamental trigonometric splines. Boundary conditions are incorporated by prescribing specific values for the corresponding spline parameters. The theoretical exposition is supplemented with illustrative examples.

It should be noted that incomplete trigonometric splines are functions with a certain evenness. In our opinion, in many cases this property is convenient when setting boundary conditions for the desired solution. The given method assumes generalization. Thus, in particular, the minimization of the residual can be carried out by the method of discrete least squares and its variations. This method can also be applied to linear differential equations of arbitrary orders.

Keywords: *fundamental trigonometric splines, incomplete trigonometric splines, even fundamental splines, odd fundamental splines, boundary value problems, linear differential equations.*

AMS 2020 classification: 65D05, 41A05.

Introduction

The primary objective of this study is to develop a methodology for constructing approximate solutions to the first boundary value problem for linear differential equations, utilizing incomplete trigonometric splines of both even and odd types. Linear differential equations frequently arise in scientific and engineering applications. Although a variety of theoretical approaches to their solution have been established, obtaining exact solutions is often challenging and, in many cases, unattainable. In such cases, the problem of finding approximate solutions to such equations is considered; such solutions are found using numerical methods.

Numerical methods for finding approximate solutions to differential equations can be divided into two classes: numerical and numerical-analytical methods.

Numerical methods allow us to find discrete values of the desired approximate solution on the interval of the differential equation. There are many such methods known, among which we mention the Runge – Kutta, Adams, Euler, Krylov, etc. methods and their modifications. Such methods are quite simple to implement on modern computers and, if applicable, lead to satisfactory results.

When applying numerical-analytical methods, an approximate solution is sought in the form of some sufficiently smooth function that depends on several parameters; these parameters are chosen based on certain conditions. Since we will be dealing with methods of this class in the future, let's consider the use of such methods in more detail.

Suppose we have a linear differential operator of the 2nd order of the form

$$L_2 u = u''(x) + p_1(x)u'(x) + p_2(x)u(x), \quad (1)$$

where $x \in [a, b]$, $p_1(x), p_2(x)$ are some functions that are generally continuous on $[a, b]$.

Consider the equation

$$L_2 u = f(x). \quad (2)$$

It is known from the theory of differential equations that the general solution $u(x)$ of this equation contains 2 arbitrary constants. To determine these arbitrary constants, additional 2 conditions are added to (2). We will consider the case when these conditions are imposed on the values of the desired solution at points a and b of the segment $[a, b]$, and have the form

$$u(a) = a_0; u(b) = b_0. \quad (3)$$

Such conditions are called boundary conditions and are said to be the first boundary value problem.

Let $C_{[a,b]}^k$ be the class of functions that have a continuous derivative of order k on $[a, b]$. Note that the classes $C_{[a,b]}^0$ contain continuous functions, and the classes $C_{[a,b]}^{-1}$ contain piecewise continuous functions with a finite number of discontinuities of the first kind.

Let's look for a solution to this equation along with the boundary conditions in the form of a function $\varphi(x, \alpha_1, \alpha_2, \dots, \alpha_N)$ that satisfies the conditions:

$$\text{a) } \varphi(x, \alpha_1, \alpha_2, \dots, \alpha_N) \in C_{[a,b]}^r, \text{ where } r \geq 2;$$

$$\text{b) } \varphi(a, \alpha_1, \alpha_2, \dots, \alpha_N) = a_0; \varphi(b, \alpha_1, \alpha_2, \dots, \alpha_N) = b_0;$$

and $\alpha_1, \alpha_2, \dots, \alpha_N$ are some unspecified parameters ($N = 1, 2, \dots$).

Substituting the function $\varphi(x, \alpha_0, \alpha_1, \dots, \alpha_N)$ into (2), we get a residual

$$L_n \varphi(x, \alpha_0, \alpha_1, \dots, \alpha_N) - f(x) = \varepsilon(x, \alpha_0, \alpha_1, \dots, \alpha_N). \quad (4)$$

By minimizing the residual in one sense or another, it is easy to obtain the conditions for determining the parameters of $\alpha_0, \alpha_1, \dots, \alpha_N$.

There are various methods for minimizing the residual: the collocation method, the least squares method (continuous and discrete variants), the method of moments and its variants, the subdomain method (Fletcher, 2012), etc. In this paper, without losing generality, we will limit ourselves to the collocation method. As is known, this method consists of the following.

A certain system of points $a \leq x_1 < x_2 < \dots < x_N \leq b$ is set on $[a, b]$ and it is required that the following conditions be met

$$\varepsilon(x_i, \alpha_1, \alpha_2, \dots, \alpha_N) = 0 \quad (i = 1, 2, \dots, N). \quad (5)$$

From this system of N equations, it is possible to determine N parameters $\alpha_0, \alpha_1, \dots, \alpha_N$.

When applying numerical-analytical methods for constructing approximate solutions, the method of constructing the function $\varphi(x, \alpha_1, \alpha_2, \dots, \alpha_N)$ plays an important role. In many cases, a generalized polynomial is chosen as this function

$$P_N(x) = \alpha_1 \omega_1(x) + \alpha_2 \omega_2(x) + \dots + \alpha_N \omega_N(x), \quad (6)$$

where $\omega_1(x), \omega_2(x), \dots, \omega_N(x)$ is a system of linearly independent functions.

In the last decades, splines have been widely chosen as such systems of functions. The fundamental basis of spline theory and its application to variational problems was established in the classical works of (Ahlberg, Nilson, & Walsh, 1967; Prenter, 1975; Schumaker, 2010). Significant contributions to the development of spline collocation methods for differential equations were also made by (Fairweather, & Meade, 1989). However, polynomial splines have limitations when approximating functions with oscillating nature. For such problems, the use of trigonometric splines, first introduced by (Schoenberg, 1964), is more natural. The fundamental theory of functions on grids, which is essential for such approximations, is detailed in (Guter, Kudryavtsev, & Levitan, 1966).

Recently, there has been a resurgence of interest in using trigonometric B-splines for solving boundary value problems. (Abbas et al., 2014) and (Nazir et al., 2016) demonstrated the unconditional stability and effectiveness of cubic trigonometric B-splines collocation for diffusion and advection problems. (Heilat, 2023) extended this to systems of second-order boundary value problems, while (Yaseen, Abbas, & Riaz, 2021) applied it to fractional diffusion equations. To improve accuracy, researchers have introduced hybrid and extended B-splines with free parameters, as seen in the work of (Khan, & Misro, 2025), where optimization algorithms are used to tune shape parameters. Advanced applications to nonlinear and singular problems have been explored by (Kukreja, & Shallu, 2025; Roul, 2024), while (Zou, 2025) proposed optimization-based trigonometric interpolation for mixed boundary conditions.

Despite these advances, most B-spline methods use control points as coefficients, which complicates the satisfaction of boundary conditions (3). In this study, we propose a method based on incomplete fundamental trigonometric splines. Classes of such splines were introduced by (Denysiuk, 2018), and their representation forms were analyzed by (Denysiuk, & Rybachuk, 2024). A distinctive feature of our method is that the coefficients α_k in (6) are the values of the solution itself. This allows boundary conditions (3) to be satisfied trivially by setting $\alpha_1 = a_0$ and $\alpha_N = b_0$, significantly simplifying the computational algorithm.

1. Main definitions and results

In (Denysiuk, & Rybachuk, 2024), three forms of representation of trigonometric splines were given: through discrete (finite) Fourier series, through trigonometric B-splines, and through fundamental trigonometric splines.

When developing methods for constructing approximate solutions to boundary value problems for linear differential equations based on trigonometric splines, the form of representation of trigonometric splines and the interpretation of the parameters of trigonometric splines in each case play an important role. Thus, in the case of representation of trigonometric splines through discrete (finite) Fourier series, discrete Fourier coefficients act as defining parameters; both periodic and non-periodic splines can be constructed.

In the case of representing splines through trigonometric B-splines, the defining parameters are the coefficients at these B-splines, which are difficult to interpret in any reasonable way; at the same time, only periodic splines are currently plotted.

Finally, in the case of representing trigonometric splines through fundamental splines, the determining parameters are the values of the desired solution at the nodes of the given collocation grids; both periodic and non-periodic splines can be constructed.

Although all three forms can be used for methods of constructing approximate solutions, in our opinion, the most convenient form is the one using fundamental trigonometric splines. This is because using the values of the desired solution as parameters greatly simplifies the enforcement of the boundary conditions for this solution. Therefore, in the following, we will limit ourselves to considering the case when the desired solution is constructed using fundamental trigonometric splines.

Let $[a, b]$ be a certain segment of the real axis, and let there be a certain grid $\Delta_N = \{x_j\}_{j=1}^N$ on this segment, and $a \leq x_1 < x_2 < \dots < x_N \leq b$. The system of functions $\varphi_k(x)$ ($k = 1, 2, \dots, N$) on the grid Δ_N is called fundamental if the following conditions are met (Guter, Kudryavtsev, & Levitan, 1966).

$$\varphi_k(x_j) = \begin{cases} 1, k = j; \\ 0, k \neq j, k, j = 1, 2, \dots, N. \end{cases} \quad (7)$$

Given a system of fundamental functions, the function $\Phi_N(x)$ interpolating a given function $f(x)$ at the nodes of the grid Δ_N can be represented as

$$\Phi_N(x) = \sum_{k=1}^N f(x_k) \varphi_k(x).$$

Similarly, systems of fundamental polynomial splines are introduced.

Systems of trigonometric fundamental splines are of three types: complete fundamental splines, as well as even and odd fundamental splines (Denysiuk, & Rybachuk, 2024).

Complete fundamental splines are periodic functions and can be used to find periodic solutions of linear differential equations if such solutions exist; recall that the existence of such solutions is discussed. In this paper, we will not consider the problem of finding periodic solutions of differential equations.

Even and odd fundamental splines can be used to find nonperiodic solutions to differential equations. Let us consider this issue in more detail.

Let us define uniform grids $\Delta_2^{(I)} = \{x_j^{(I)}\}_{j=1}^N$ on $[0, \pi]$, where $x_j^{(0)} = \frac{\pi}{N-1}(j-1)$, $x_j^{(1)} = \frac{\pi}{2N}(2j-1)$ and $\Delta_3^{(I)} = \{x_j^{(I)}\}_{j=1}^N$, where $x_j^{(0)} = \frac{\pi}{N+1}j$, $x_j^{(1)} = \frac{\pi}{2N}(2j-1)$, I is the grid indicator ($I = 0, 1$), and $N = 1, 2, 3, \dots$. We will not introduce separate notations for the nodes of these grids, since in each case we will determine which grid we are talking about.

Non-periodic even and odd trigonometric splines are constructed on the grids $\Delta_2^{(I)}$ and $\Delta_3^{(I)}$, respectively (Denysiuk, & Rybachuk, 2024). Even and odd trigonometric splines have different properties; in particular, even trigonometric splines take non-zero values at the points 0 and π , and their derivatives take zero values at these points. Odd splines, on the other hand, take on zero values at the points 0 and π , and their derivatives at these points take on non-zero values. Taking this into account, and wanting to deal with uniform grids, we will limit ourselves to considering even splines with the same stitching and interpolation grids $\Delta_2^{(0)}$ and odd splines with the same stitching and interpolation grids $\Delta_3^{(1)}$.

Here are the expressions for constructing fundamental even splines on the grid $\Delta_2^{(0)}$; here and further, in the notation of even and odd fundamental splines, we will omit the indices of the stitching and interpolation grids, as well as the index of the type of convergence factors, which is equal to $\sigma 1(r, j) = j^{-(1+r)}$ (r, j are natural numbers).

The paired fundamental splines $stc(r, q, k, t)$ of order r ($r = 1, 2, \dots$), and their derivatives of order q ($q = 1, 2, \dots, r-1$) on the grid $\Delta_2^{(0)}$ are calculated by the expressions

$$stc(r, q, k, t) = \frac{2}{N-1} \left[0.5 \cdot I(q) + \sum_{j=1}^{N-2} \frac{c(\sigma 1, r, q, j, t) \cos(x_k^{(0)})}{H(\sigma 1, r, j)} + 0.5 \frac{c(\sigma 1, r, q, N-1, t) \cos(x_k^{(0)})}{H(\sigma 1, r, N-1)} \right], \quad (8)$$

where

$$I(q) = \begin{cases} 1, \text{if } q = 0, \\ 0, \text{otherwise} \end{cases}$$

$$H(\sigma 1, r, k) = \sigma 1(r, k) + \sum_{m=1}^{\infty} [\sigma 1(r, 2m(N-1) + k) + \sigma 1(r, 2m(N-1) - k)],$$

$$c(\sigma 1, r, q, j, t) = \sigma 1(r, j) j^q \cos(jt + q \frac{\pi}{2}) +$$

$$+ \sum_{m=1}^{\infty} \left[\sigma 1(r, 2m(N-1) + j) \left(2m(N-1) + j \right)^q \cos((2m(N-1) + j)t + q \frac{\pi}{2}) + \right. \\ \left. + \sigma 1(r, 2m(N-1) - j) \left(2m(N-1) - j \right)^q \cos((2m(N-1) - j)t + q \frac{\pi}{2}) \right].$$

Graphs of paired trigonometric splines are shown on the Fig. 1 ($r = 1$ and $N = 9$ for clarity). Note that here and in the following graphs, the vertical lines coincide with the nodes of the grid $\Delta_2^{(0)}$.

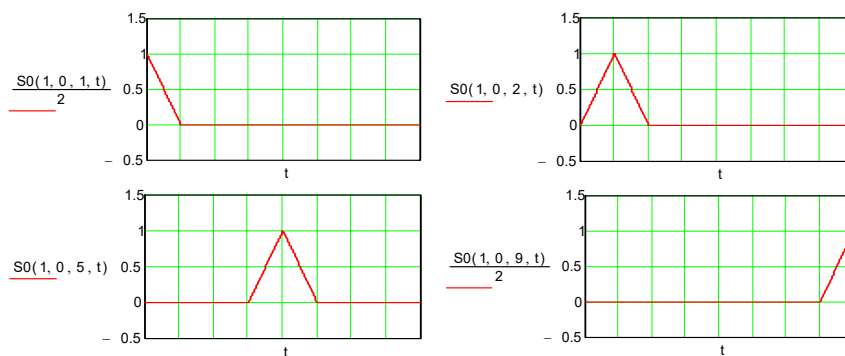


Fig. 1. Paired fundamental splines on the grid $\Delta_2^{(0)}$; $r = 1$, $N = 9$

Similarly, the odd fundamental splines $sts^{(0)}(r, q, k, t)$ of order r ($r = 1, 2, \dots$), and their derivatives of order q ($q = 1, 2, \dots, r-1$) on the grid $\Delta_3^{(0)}$, are calculated by the expression

$$sts^{(0)}(r, q, k, t) = \frac{2}{N+1} \sum_{j=1}^N \frac{s(\sigma 1, r, q, j, t) \sin(jx_k^{(0)})}{H(\sigma 1, r, j)}, \quad (9)$$

where

$$\begin{aligned} H(\sigma 1, r, k) &= \sigma 1(r, k) + \sum_{m=1}^{\infty} [\sigma 1(r, 2m(N+1) + k) + \sigma 1(r, 2m(N+1) - k)], \\ s(\sigma 1, r, q, j, t) &= \sigma 1(r, j) j^q \sin(jt + q \frac{\pi}{2}) + \\ &+ \sum_{m=1}^{\infty} (-1)^m [\sigma 1((r, 2m(N+1) + j)) ((2m(N+1) + j)^q \sin((2m(N+1) + j)t + q \frac{\pi}{2}) - \\ &- \sigma 1((r, 2m(N+1) - j)) ((2m(N+1) - j)^q \sin((2m(N+1) - j)t + q \frac{\pi}{2}))]. \end{aligned}$$

As before, we present the graphs of odd trigonometric splines (Fig. 2), assuming $r = 1$, $N = 9$. Note that the vertical lines in these graphs coincide with the nodes of the grid $\Delta 3_N^{(0)}$.

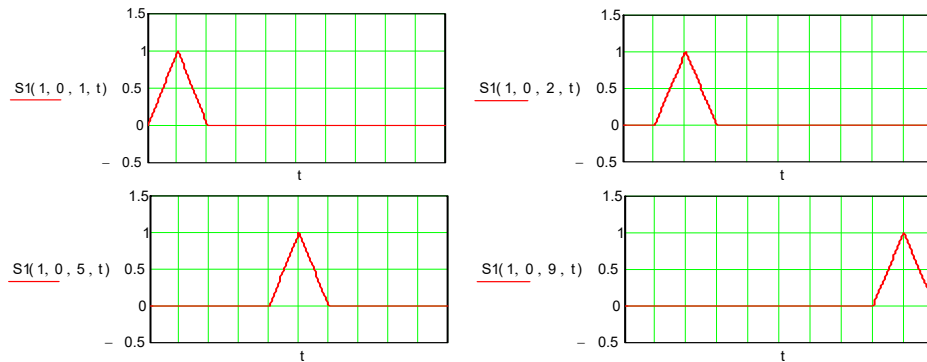


Fig. 2. Odd fundamental splines $sts^{(0)}(1, 0, k, t)$ on the grid $\Delta 3_N^{(0)}$; $r = 1$, $N = 9$

On the grid $\Delta 3_N^{(1)}$, the odd fundamental splines $sts^{(1)}(r, q, k, t)$ of order r ($r = 1, 2, \dots$), and their derivatives of order q ($q = 1, 2, \dots, r-1$), are calculated by the expressions

$$sts^{(1)}(r, q, k, t) = \frac{2}{N} \left[\sum_{j=1}^{N-1} \frac{s(\sigma 1, r, q, j, t) \sin(jx_k^{(1)})}{H(\sigma 1, r, j)} + 0.5 \frac{s(\sigma 1, r, q, N, t) \sin(Nx_k^{(1)})}{H(\sigma 1, r, N)} \right], \quad (10)$$

where

$$\begin{aligned} H(\sigma 1, r, k) &= \sigma 1(r, k) + \sum_{m=1}^{\infty} [\sigma 1(r, 2mN + k) + \sigma 1(r, 2mN - k)], \\ s(\sigma 1, r, q, j, t) &= \sigma 1(r, j) j^q \sin(jt + q \frac{\pi}{2}) + \\ &+ \sum_{m=1}^{\infty} (-1)^m [\sigma 1(r, 2mN + j) (2mN + j)^q \sin((2mN + j)t + q \frac{\pi}{2}) - \\ &- \sigma 1(r, 2mN - j) (2mN - j)^q \sin((2mN - j)t + q \frac{\pi}{2})]. \end{aligned}$$

Graphs of odd trigonometric splines are shown on the Fig. 3, assuming $r = 1$, $N = 9$. Note that the vertical lines in these graphs coincide with the nodes of the grid $\Delta 3_N^{(0)}$.

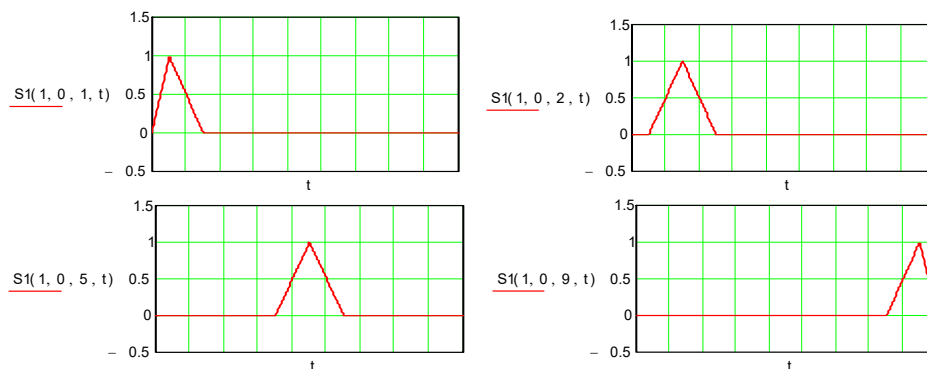


Fig. 3. Odd fundamental splines $sts^{(1)}(1, 0, k, t)$ on the grid $\Delta 3_N^{(0)}$; $r = 1$, $N = 9$

Given the even and odd fundamental splines $stc(r, q, k, t)$, $sts^{(0)}(1, 0, k, t)$ and $sts^{(1)}(1, 0, k, t)$, the splines $Stc(r, q, t)$ and $Sts(r, q, t)$ interpolating a continuous function $f(x)$ at the nodes of the grids $\Delta 2_N^{(0)}$, $\Delta 3_N^{(0)}$ and $\Delta 3_N^{(1)}$ can be represented as

$$Stc(r, q, t) = 0.5f(x_1)stc(r, q, 1, t) + \sum_{k=2}^{N-1} f(x_k)stc(r, q, k, t) + 0.5f(x_N)stc(r, q, N, t), \quad (11)$$

$$Sts^{(0)}(r, q, t) = \sum_{k=1}^N f(x_k)sts^{(0)}(r, q, k, t), \quad (12)$$

$$Sts^{(1)}(r, q, t) = \sum_{k=1}^N f(x_k)sts^{(1)}(r, q, k, t). \quad (13)$$

It is clear that interpolation splines $Stc(r, q, k, t)$, $Sts^{(0)}(r, q, k, t)$, $Sts^{(1)}(r, q, k, t)$ and fundamental splines $stc(r, q, k, t)$, $sts^{(0)}(r, q, k, t)$, $sts^{(1)}(r, q, k, t)$ belong to the class $C_{[0, \pi]}^{r-1}$.

Let us now consider the first boundary value problem for (2) with boundary conditions (3), which, as we have already mentioned, is formulated as follows. Find the function $u(x), u(x) \in C_{[0, \pi]}^r$ ($r \geq 2$), $x \in [0, \pi]$, which satisfies the equation

$$u''(x) + p_1(x)u'(x) + p_2(x)u(x) = f(x) \quad (14)$$

and boundary conditions

$$u(0) = a_0; u(\pi) = b_0. \quad (15)$$

There are three ways to find an approximate solution $u^*(x)$ to this problem.

a) On the grid $\Delta 2_N^{(0)}$ via paired splines

$$u^*(x) = 0.5\alpha_1 stc(r, q, 1, x) + \sum_{k=2}^{N-1} \alpha_k stc(r, q, k, x) + 0.5\alpha_N stc(r, q, N, x), \quad (16)$$

where $r \geq 3$ and the parameters α_k ($k = 2, 2, \dots, N-1$) are determined from the collocation conditions.

It is necessary to take into account the fact that when finding an approximate solution in the form of (16), additional boundary conditions of the form $u'(0) = u'(\pi) = 0$ are also fulfilled. This is due to the fact that when differentiating (16), the functions $\cos(x)$ are transformed into functions $-\sin(x)$, which turn to 0 at the ends of the segment $[0, \pi]$.

b) On the grid $\Delta 3_N^{(0)}$ through odd splines in the form

$$u^{*(0)}(x) = \sum_{k=1}^N \alpha_k sts^{(0)}(r, q, k, x), \quad (17)$$

and on the grid $\Delta 3_N^{(1)}$ in the form of

$$u^{*(1)}(x) = \sum_{k=1}^N \alpha_k sts^{(1)}(r, q, k, x), \quad (18)$$

where $r \geq 3$, and the parameters α_k ($k = 1, 2, \dots, N$) are determined from the collocation conditions. Note that such splines are conveniently used for problems with zero boundary conditions.

In cases where the solution to the equation is sought on the segment $[a, b]$, this segment is mapped to the segment $[0, \pi]$ by linearly replacing the variables.

Consider the case when the solution to problems (14), (15) is sought in the form (17). It is clear that in this case, the nodes $x_1 = 0$ and $x_N = \pi$ of the $\Delta 2_N^{(0)}$ mesh coincide with the ends of the segment $[0, \pi]$, and, therefore, it follows from (15) that $\alpha_1 = a_0$ and $\alpha_N = b_0$. To determine the parameters of $\alpha_2, \alpha_2, \dots, \alpha_{N-1}$, we substitute (17) into (14) and equate the values of the left and right parts at the nodes $x_2, x_2, \dots, x_{N-1}$ of the grid $\Delta 2_N^{(0)}$; as a result, we obtain the following system of linear algebraic equations with respect to the desired parameters $\alpha_2, \alpha_2, \dots, \alpha_{N-1}$:

$$0.5a_0 A(r, 1, x_j) + \sum_{k=2}^{N-1} \alpha_k A(r, k, x_j) + 0.5b_0 A(r, N, x_j) = f(x_j) \quad (j = 2, 3, \dots, N-1), \quad (19)$$

where

$$A(r, k, x_j) = stc(r, 2, k, x_j) + p_1(x_j)stc(r, 1, k, x_j) + p_2(x_j)stc(r, 0, k, x_j) \quad (k = 1, 2, \dots, N; j = 2, 3, \dots, N-1).$$

2. Examples

To illustrate these provisions, we will provide examples.

Example 1. Let us consider a boundary value problem

$$u''(x) + \frac{C}{1+x} u'(x) - \frac{x}{1+x} u(x) = \frac{C-2-x^2(1+x)}{(1+x)^3}, \quad x \in [0, 1];$$

$$u(0) = 0; u(1) = 0.5, \quad (C = 0, 1, 10).$$

The exact solution to this problem is

$$u_0(x) = \frac{x}{1+x}.$$

By composing a system of equations of the type (19) at $r = 3$, and solving it, we obtain approximate solutions $u_1(x)$ of this problem for the values $C = 0, 1, 10$. Graphs of exact solution and approximate solutions for different values of the parameter C are shown on the Fig. 4.

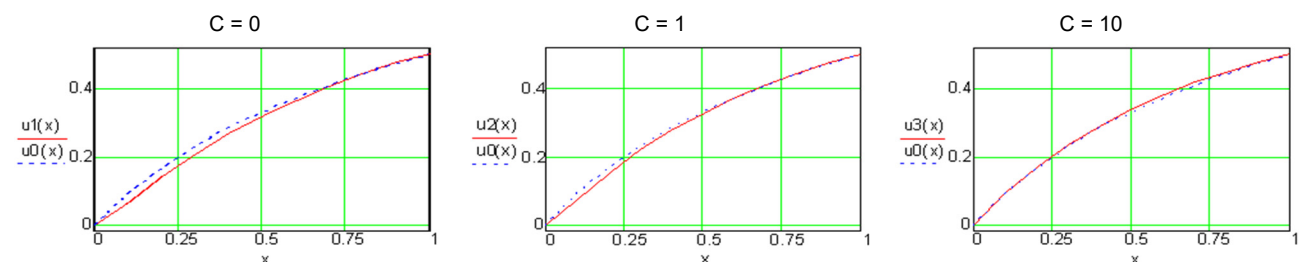


Fig. 4. The exact solution $u_0(x)$ and approximate solutions $u_1(x)$, $u_2(x)$ and $u_3(x)$, corresponding to the values of the parameter $C = 0, 1, 10$

Note that the largest deviations of the exact solution from the approximate solution do not exceed 0.048 when $C = 0$, 0.044 when $C = 1$, and 0.042 when $C = 10$.

Example 2. Let us consider a boundary value problem

$$u''(x) + u(x) = \cos(x) \cos(2x), \quad x \in [0, \pi];$$

$$u(0) = 1; u(\pi) = -1.$$

The exact solution to this problem is

$$u_0(x) = 1.0625 \cos(x) - 0.4 \sin(x) - 0.0625 \cos(3x) + 0.25x \sin(x).$$

By composing systems of equations of the type (19) at $r = 3, 4, 5$, and solving them, we obtain approximate solutions $u_3(x)$, $u_4(x)$, $u_5(x)$ of this problem corresponding to the given values of the parameter r . Graphs of exact solution and approximate solutions for different values of the parameter r are shown on the Fig. 5.

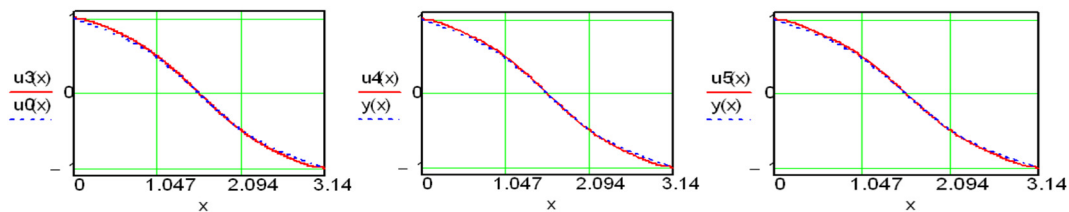


Fig. 5. The exact solution $u_0(x)$ and approximate solutions $u_3(x)$, $u_4(x)$ and $u_5(x)$, corresponding to the values of the parameter $r = 3, 4, 5$

The largest deviations of the exact solution from the approximate solution do not exceed 0.048 at $r = 3$, 0.045 at $r = 4$, and 0.043 at $r = 5$.

Let us now consider the case when the solution to problems (14), (15) is sought in the form (18), with (15) having the form

$$u(0) = 0; u(\pi) = 0. \quad (20)$$

It is clear that the nodes x_1 and x_N of the $\Delta 3_N^{(0)}$ mesh do not coincide with the ends of the segment $[0, \pi]$. Since (18) is equal to zero at points 0 and π , the boundary conditions are automatically satisfied. By drawing up a system of equations of the form

$$\sum_{k=1}^N \alpha_k A(r, k, x_j) = f(x_j), \quad (j = 2, 3, \dots, N-1), \quad (21)$$

where

$$A(r, k, x_j) = sts^{(0)}(r, 2, k, x_j) + p_1(x_j)stc^{(0)}(r, 1, k, x_j) + p_2(x_j)stc^{(0)}(r, 0, k, x_j), \\ (k = 1, 2, \dots, N; j = 1, 2, 3, \dots, N)$$

and finding the value of α_k ($k = 1, 2, \dots, N$), we construct an approximate solution $u^{*(0)}(x)$.

Reasoning similarly, and using the splines $sts^{(1)}(r, q, k, x)$ to construct the values $A(r, k, x_j)$, it is easy to construct the solution $u^{*(1)}(x)$.

As before, to illustrate the above provisions, we will provide an example.

Example 3. Let us consider a boundary value problem

$$u''(x) + u(x) = -x, \quad x \in [0, \pi] \quad (x \in [0, 1]), \\ u(0) = 0; u(1) = 0.$$

The exact solution to this problem is

$$u(x) = \frac{\sin(x)}{\sin(1)} - t.$$

By composing systems of equations of the type (21) at $r = 3, 4, 5$, and solving them, we obtain approximate solutions $u_3^{(0)}(x)$, $u_4^{(0)}(x)$, $u_5^{(0)}(x)$ of this problem corresponding to the given values of the parameter r . Graph of exact $u(x)$ and approximate $u_4^{(0)}(x)$ solutions to this problem is shown on the Fig. 6; we also show the errors of other solutions for different values of the parameter r .

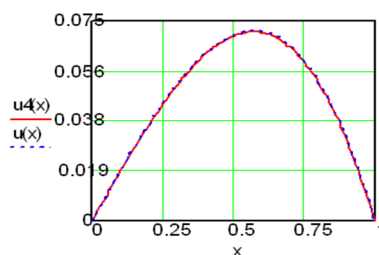


Fig. 6. The exact solution $u(x)$ and the approximate solution $u_4^{(0)}(x)$, corresponding to the values of the parameter $r = 4$

The largest deviations of the exact solution from the approximate solution do not exceed 0.00157 at $r = 3$, 0.00105 at $r = 4$, and 0.00098 at $r = 5$.

Similar calculations were performed for the approximate solutions $u_3^{(1)}(x)$, $u_4^{(1)}(x)$, $u_5^{(1)}(x)$; the largest deviations of the exact solution from the approximate solution do not exceed 0.0005 at $r = 3, 4$, but increase to 0.00063 at $r = 5$.

Discussion and conclusions

The study introduces methodologies for constructing approximate solutions to the first boundary value problem associated with linear differential equations through the application of incomplete trigonometric splines of both even and odd forms.

The utilization of even trigonometric splines is particularly appropriate in cases where the first derivative of the solution is subject to homogeneous boundary conditions, whereas odd trigonometric splines are recommended when the solution itself satisfies zero boundary conditions. To substantiate the theoretical framework, illustrative examples are provided.

Furthermore, it is demonstrated that the computational error exhibits dependence on the parameter r , which encapsulates the differential properties inherent to the employed trigonometric splines.

The proposed approaches underscore the necessity for continued investigation to refine and extend the methods for approximating solutions to the first boundary value problem in the context of linear differential equations.

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НЕПОВНІ ТРИГОНОМЕТРИЧНІ СПЛАЙНИ В ЗАДАЧАХ ПОБУДОВИ НАБЛИЖЕНИХ РОЗВ'ЯЗКІВ ЛІНІЙНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ДРУГОГО ПОРЯДКУ

Представлено числовий аналітичний метод для отримання наближених розв'язків першої крайової задачі, пов'язаної з лінійними диференціальними рівняннями другого порядку. Підхід базується на використанні неповних фундаментальних тригонометричних сплайнів, як парних, так і непарних, структурні властивості яких дуже схожі на властивості класичних поліноміальних сплайнів та були широко досліджені в попередніх роботах. Запропонована методологія використовує мінімізацію залишків за допомогою встановленого методу колокації, тим самим зводячи вихідну задачу до розв'язання систем лінійних алгебраїчних рівнянь відносно параметрів, що визначають сплайн. Відмінна особливість цього методу полягає в тому, що самі значення шуканого розв'язку є визначальними параметрами фундаментальних тригонометричних сплайнів. Гранічні умови враховано шляхом призначення конкретних значень для відповідних параметрів сплайна. Теоретичний виклад доповнено ілюстративними прикладами.

Варто зазначити, що неповні тригонометричні сплайни є функціями з певною парністю. На нашу думку, у багатьох випадках ця властивість зручна під час задання граничних умов для шуканого розв'язку. Наведений метод передбачає узагальнення. Зокрема, мінімізацію залишку можна здійснити методом дискретних найменших квадратів і його варіаціями. Цей метод також можна застосовувати до лінійних диференціальних рівнянь довільного порядку.

Ключові слова: фундаментальні тригонометричні сплайни, неповні тригонометричні сплайни, парні фундаментальні сплайни, непарні фундаментальні сплайни, крайові задачі, лінійні диференціальні рівняння.

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