

ДИФЕРЕНЦІАЛЬНІ РІВНЯННЯ, МАТЕМАТИЧНА ФІЗИКА ТА МЕХАНІКА

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ROBIN'S PROBLEM FOR A FOURTH-ORDER HYPERBOLIC EQUATION

The Robin problem, also known as a mixed boundary condition, is often applied to partial differential equations. It defines a relationship between the value of a function and its derivative at the boundary of a domain, enabling the modeling of physical processes that affect both the properties of the surface itself and the flow of matter or energy. Fourth-order hyperbolic equations describe complex processes involving wave propagation and their interactions, enabling the modeling of dispersion effects often present in real-world physical processes. Such equations are used in hydrodynamics to model wave propagation at great depths, where both the velocity and the influence of dispersion on wave behavior are taken into account; in elasticity theory to analyze deformations and wave propagation in elastic materials, where dispersion effects and complex boundary conditions are present; and in seismology.

The aim of this work is to solve the Robin problem for a fourth-order hyperbolic equation with conditions imposing a relationship between certain derivatives of the function at the boundary. For this problem, the obtained solution to the Cauchy problem for a second-order hyperbolic equation is used. The solution to the Cauchy problem is obtained using differential operators that transform arbitrary functions into a regular solution of an m -th order hyperbolic equation. By reducing the problem to a solved Cauchy problem for a second-order hyperbolic equation and matching the boundary conditions of these problems, a system of equations arises whose solution exists and is unique as a solution to the Volterra integral equation and as a solution to the inhomogeneous Bessel equation. The solution to the Robin problem is a combination of the obtained solutions to the corresponding Cauchy problems for second-order equations.

Keywords: differential operator, regular solutions, iterated equations of hyperbolic type, Robin's problem.

Classification according to AMS 2020: 35Q74, 35R09.

Introduction

Hyperbolic equations are used to describe wave propagation processes, such as sound, heat, and light waves, as well as in electromagnetism and elasticity theory. They are used to model with high accuracy dynamic phenomena characterized by wave propagation and interaction, as well as the vibration of bridges, structures, and mechanical systems. Mathematical models are formulated as boundary value problems for partial differential equations in bounded and unbounded domains, for which a general theory of function-theoretical methods is considered in (Gilbert, 1969).

In the Riquier and Robin problems, which consider mixed boundary conditions for partial differential equations, solutions are determined taking into account the values of the function and its derivative at the boundary of the domain. Problems for higher-order elliptic equations are considered in (Gomez-Polanco, Guevara, & Molina, 2013; Karachik, 2003). In the Robin problem (Rozkosz, 2025), the existence of a positive renormalized solution was proven and a stochastic representation for a uniformly elliptic divergence operator with a measure on the right-hand side of the equation and an absorbing term on the boundary containing collapsing terms was proposed. In the paper (Guarino Lo Bianco, La Manna, & Velichkov, 2021), a two-phase free limit problem was studied for the first time, in which the solution satisfies the Robin limit condition. The case in which the solution is continuous on the free boundary was considered, and the existence and regularity of minimizers for the associated variational problem were proven.

The use of fourth-order hyperbolic equations allows one to model dispersion effects, which are often present in real physical processes. Wave propagation in dispersive media (liquids, gases, and certain types of solids) requires the formulation of fourth-order hyperbolic equations that take this dependence into account.

This paper is devoted to deepening the results of studying solutions to differential equations containing operators of the

form $D_m = \frac{\partial^m}{\partial x_1 \partial x_2 \dots \partial x_m}$ and their iterations, taking into account the Robin limit condition. Methods for solving such equations

are based on the creation of differential and integral operators that determine the solution of hyperbolic and elliptic equations and systems.

The differential operators constructed in (Lyashko et al., 2024) that transform arbitrary functions of the domain $D \subset \mathbb{R}^2$ into a regular solution to the Robin problem equation

$$Y_k^m W = \left(\frac{\partial^2}{\partial x \partial y} + \frac{k}{2(x+y)} \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x} \right) \right)^m W = 0, m \in N, k/2 \in Z, \quad (1)$$

are applicable to solving the Robin problem.

1. Main results

Consider the differential equation

$$W_{xy} + \frac{(n-m)\phi'(x)}{\phi(x)+\psi(y)} W_y - \frac{n(m+1)\psi'(y)}{(\phi(x)+\psi(y))^2} W = 0, \quad (2)$$

where $\phi(x), \psi(y)$ are functions satisfying the condition $(\phi(x)+\psi(y))\phi'(x)\psi'(y) \neq 0$, $n, m \in N \cup \{0\}$, and $W(x, y)$ are defined by the equality (Lyashko et al., 2024)

$$W(x, y) = \sum_{k=0}^n (-1)^{n-k} \frac{n!(m+1)_{n-k}}{k!(n-k)!} \frac{(\phi'(x))^{n-k}}{(\phi(x)+\psi(y))^{n-k}} g^{(k)}(x) + \sum_{k=0}^m (-1)^{m-k} \frac{m!(n+1)_{m-k}}{k!(m-k)!} \frac{(\psi'(y))^{m-k}}{(\phi(x)+\psi(y))^{m-k}} f^{(k)}(y), \quad (3)$$

is a solution to equation (2), and the functions $g(x), f(y)$ are arbitrary, differentiable a sufficient number of times in the domain D .

Let us formulate the *Cauchy problem*. In the domain $D = [a, b] \times [a, b]$, find a solution of the equation

$$W_{xy} - \frac{2}{(x+y)^2} W = 0, \quad (4)$$

satisfying the conditions

$$W(x, y)|_{y=x} = \alpha(x), \left(\frac{\partial W}{\partial x} - \frac{\partial W}{\partial y} \right)|_{y=x} = \beta(x), \quad (5)$$

where $\alpha(x) \in C^2([a, b]), \beta(x) \in C^2([a, b])$ are given functions, $x+y \neq 0$ in the domain D .

According to formula (3), satisfying conditions (5), the solution to the Cauchy problem will have the form

$$W(x, y) = (\alpha' - \beta) \frac{x(y-x)}{x+y} + \alpha(x) + \frac{1}{x+y} \int_x^y (y^2 - (y-x)t - t^2) \gamma(t) dt, \quad (6)$$

where $\gamma(t) = \frac{1}{2t}(\alpha'(t) - \beta(t)) + \frac{1}{2}(\alpha'(t) - \beta'(t)) - \frac{1}{2t^2}\alpha(t)$.

We move on to equation (1) for $2k \in Z$. Replacing $W = (x+y)^{n+1}V$ into equation (1), takes the form

$$Y_n^m V = \left(\frac{\partial^2}{\partial x \partial y} + \frac{n(n+1)}{(x+y)^2} \right)^m V = 0, m \in N, n \in N. \quad (7)$$

Let D be the domain in the upper half-plane. Using mathematical induction, it was established (Lyashko et al., 2024) that the solution to equation (7) is

$$V(x, y) = \sum_{i=0}^{m-1} (x+y)^i V_i(x, y), \quad (8)$$

where $V_i(x, y)$, $i = \overline{0, m-1}$, are $2(i+1)$ continuously differentiated solutions of the equation $Y_n V = 0$.

Theorem 1. For any solution of the equation $Y_n^m V = 0$, $m \geq 2$, in the domain D there are functions $g_i(x), f_i(y)$ such that

$$V(x, y) = \sum_{i=0}^{m-1} (x+y)^i \times \left(\sum_{k=0}^n (-1)^{n-k} \frac{(2n-k)!}{k!(n-k)!} \frac{1}{(x+y)^{n-k}} \frac{\partial^k}{\partial x^k} g_i(x) + \sum_{k=0}^n (-1)^{n-k} \frac{(2n-k)!}{k!(n-k)!} \frac{1}{(x+y)^{n-k}} \frac{\partial^k}{\partial y^k} f_i(y) \right) \quad (9)$$

and vice versa, if $g_i(x), f_i(y)$ - are arbitrary in D functions, differentiated the required number of times, then (9) is a solution of equation (7) in D .

Let us apply formula (9) to solve the Robin problem for the equation $Y^2 V = 0$.

Robin problem. Let D be a square $[a, b] \times [a, b]$. Find in the domain D a solution of the equation

$$Y^2 V = \left(\frac{\partial^2}{\partial x \partial y} - \frac{2}{(x+y)^2} \right)^2 V = 0, \quad (10)$$

справджуючи умови

$$\lim_{y \rightarrow x} \left(V + \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) V \right) = a_1(x), \quad (11)$$

$$\lim_{y \rightarrow x} \left(\left(\frac{\partial^2}{\partial x \partial y} - \frac{2}{(x+y)^2} \right) V + \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) \left(\frac{\partial^2}{\partial x \partial y} - \frac{2}{(x+y)^2} \right) V \right) = a_2(x).$$

Here $a_1(x)$, $a_2(x)$ are given functions of sufficient smoothness.

In accordance with statement (8), we seek the solution of problem (10), (11) in the form

$$V(x, y) = V_0(x, y) + (x + y)V_1(x, y).$$

Satisfying boundary conditions (11), we obtain

$$\lim_{y \rightarrow x} \left(V_0 + (x + y)V_1 + \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) (V_0 + (x + y)V_1) \right) = a_1(x),$$

$$\lim_{y \rightarrow x} \left(\left(\frac{\partial^2}{\partial x \partial y} - \frac{2}{(x + y)^2} \right) (V_0 + (x + y)V_1) + \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) \left(\frac{\partial^2}{\partial x \partial y} - \frac{2}{(x + y)^2} \right) (V_0 + (x + y)V_1) \right) = a_2(x).$$

Since V_0, V_1 satisfy to equation

$$\frac{\partial^2 V}{\partial x \partial y} - \frac{2}{(x + y)^2} V = 0,$$

then to find functions $V_0(x, y)$, $V_1(x, y)$ we have the following Cauchy problems

$$\text{I. } \frac{\partial^2 V_0}{\partial x \partial y} - \frac{2}{(x + y)^2} V_0 = 0, \quad V_0|_{y=x} = \alpha_1(x), \quad \frac{\partial V_0}{\partial x} - \frac{\partial V_0}{\partial y}|_{y=x} = \beta_1(x), \quad (12)$$

$$\text{II. } \frac{\partial^2 V_1}{\partial x \partial y} - \frac{2}{(x + y)^2} V_1 = 0, \quad V_1|_{y=x} = \alpha_2(x), \quad \frac{\partial V_1}{\partial x} - \frac{\partial V_1}{\partial y}|_{y=x} = \beta_2(x). \quad (13)$$

The functions $\alpha_1(x)$, $\beta_1(x)$, $\alpha_2(x)$, $\beta_2(x)$ are determined from the system of equations:

$$\begin{cases} \alpha_1(x) + 2x\alpha_2(x) + \beta_1(x) + 2x\beta_2(x) = a_1(x) \\ 2\alpha_2'(x) - \beta_2(x) + \frac{1}{2x^2}\beta_1(x) + \beta_2'(x) = a_2(x) \\ \alpha_1''(x) - \frac{1}{2x^2}\alpha_1(x) = \beta_1'(x) \\ \alpha_2''(x) - \frac{1}{2x^2}\alpha_2(x) = \beta_2'(x) \end{cases},$$

transforming which, we obtain

$$\begin{cases} 2x^2\alpha_2^{(4)}(x) + (4x + 2x^2)\alpha_2^{(3)}(x) + 4x\alpha_2''(x) - (1 + 1/x)\alpha_2'(x) - \frac{3}{2x^2}\alpha_2(x) = 4x\alpha_2'(x) + 2x^2\alpha_2''(x) \\ \beta_2'(x) = \alpha_2''(x) - \frac{1}{2x^2}\alpha_2(x) \\ \alpha_1''(x) + \alpha_1'(x) - \frac{1}{2x^2}\alpha_1(x) = a_1'(x) - 2x(\alpha_2'(x) + \beta_2'(x)) - 2(\alpha_2(x) + \beta_2(x)) \\ \beta_1'(x) = \alpha_1''(x) - \frac{1}{2x^2}\alpha_1(x). \end{cases}$$

Putting $\alpha_2^{(4)}(x) = \phi(x)$, for $\phi(x)$ we will have an integral Volterra equation of the 2nd kind

$$2x^2\phi(x) + \int_{x_0}^x K(x, t)\phi(t)dt = F(x),$$

where $K(x, t) = 4x + 2x^2 + 4x(x - t) - \frac{x+1}{2x}(x - t)^2 - \frac{1}{4x^2}(x - t)^3$, $F(x) = 4xa_2'(x) + 2x^2a_2''(x)$.

Then the functions $\alpha_2(x)$, $\beta_2(x)$ are expressed by $\phi(x)$:

$$\alpha_2(x) = \frac{1}{6} \int_{x_0}^x \phi(t)(x - t)^3 dt, \quad \beta_2(x) = \int_{x_0}^x \phi(t) \left(\frac{11}{24}x^2 + \frac{1}{4}(x - t) - tx + \frac{1}{4}t^2 \ln \frac{t}{x} + t^2 - \frac{t^3}{12x} \right) dt.$$

The function $\alpha_1(x)$ is a solution of the equation

$$\alpha_1''(x) + \alpha_1'(x) - \frac{1}{2x^2}\alpha_1(x) = G(x),$$

where $G(x) = a_1'(x) - \frac{3}{4}x^3\phi(x) - \int_{x_0}^x \phi(t) \left((x - t)^2 \left(x + \frac{x - t}{3} \right) + \frac{33}{12}x^2 + \frac{3}{2}t^2 + x - \frac{1}{2}t - 4tx + \frac{1}{2}t^2 \ln \frac{t}{x} \right) dt$,

which is a non-homogeneous Bessel equation, and has the form

$$\alpha_1(x) = \sqrt{x}e^{-x/2}J_{\sqrt{3}/2}\left(\frac{ix}{2}\right) \int_{x_0}^x \frac{dx_1}{\exp(x_1e^{-x_1/2}J_{\sqrt{3}/2}\left(\frac{ix_1}{2}\right))} \int_{x_0}^{x_1} \exp(\sqrt{x_2}e^{-x_2/2}J_{\sqrt{3}/2}\left(\frac{ix_2}{2}\right))G(x_2)dx_2.$$

Finally,

$$\beta_1(x) = a_1(x) - \alpha_1(x) - 2x \int_{x_0}^x \phi(t) \left(\frac{1}{6}(x-t)^3 + \frac{11}{24}x^2 + \frac{1}{4}(x-t) - tx + \frac{1}{4}t^2 \ln \frac{t}{x} + t^2 - \frac{t^3}{12x} \right) dt.$$

The solution of each Cauchy problem (12), (13), according to the solved problem (4), (5), will be

$$V_0(x, y) = (\alpha'_1(x) - \beta_1(x)) \frac{x(y-x)}{x+y} + \alpha_1(x) + \frac{1}{x+y} \int_x^y (y^2 - (y-x)t - t^2) \gamma_1(t) dt, \quad (14)$$

$$\text{where } \gamma_1(t) = \frac{1}{2t}(\alpha'_1(t) - \beta_1(t)) + \frac{1}{2}(\alpha''_1(t) - \beta'_1(t)) - \frac{1}{2t^2}\alpha_1(t),$$

$$V_1(x, y) = (\alpha'_2(x) - \beta_2(x)) \frac{x(y-x)}{x+y} + \alpha_2(x) + \frac{1}{x+y} \int_x^y (y^2 - (y-x)t - t^2) \gamma_2(t) dt, \quad (15)$$

$$\text{where } \gamma_2(t) = \frac{1}{2t}(\alpha'_2(t) - \beta_2(t)) + \frac{1}{2}(\alpha''_2(t) - \beta'_2(t)) - \frac{1}{2t^2}\alpha_2(t).$$

The solution of the Robin problem (10), (11) is $V(x, y) = V_0(x, y) + (x+y)V_1(x, y)$, where $V_0(x, y)$ and $V_1(x, y)$ are defined in (14) and (15).

Discussion and conclusions

The Robin problem for a fourth-order hyperbolic equation is considered and solved by reducing it to the Cauchy problem for a second-order hyperbolic equation. The solution is obtained using differential operators that transform arbitrary functions into an established, inductively regular solution to the m-th order hyperbolic equation. A theorem for solving the m-th order hyperbolic equation is formulated, which is used to find a solution to the fourth-order equation. The method proposed in this paper allows one to solve problems with complex boundary conditions.

Authors' contribution: Iryna Aleksandrovych – conceptualization; methodology; Larysa Adzhubey – relevance analysis; discussion of the theoretical foundations of the study; Andriy Ryzhenko – analysis of sources, discussion of conclusions; Viktor Lyashko – literature review.

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ЗАДАЧА РОБЕНА ДЛЯ ГІПЕРБОЛІЧНОГО РІВНЯННЯ ЧЕТВЕРТОГО ПОРЯДКУ

Задачу Робена, відому як змішану крайову умову, часто застосовують до диференціальних рівнянь у частинних похідних. Вона визначає зв'язок між значеннями функції та її похідною на межі області, що дає змогу моделювати фізичні процеси, на які впливають як властивості самої поверхні, так і потоки речовини чи енергії. Гіперболічні рівняння четвертого порядку описують складні процеси, що охоплюють поширення хвиль та їхню взаємодію, допомагають моделювати дисперсійні ефекти, які часто наявні в реальних фізичних

процесах. Такі рівняння використовують у задачах гідродинаміки для моделювання поширення хвиль на великих глибинах, де враховують як швидкість, так і вплив дисперсії на поведінку хвиль, теорії пружності для аналізу деформацій та поширення хвиль у пружних матеріалах, коли наявні ефекти дисперсії та складні граничні умови, а також у сейсмології.

Мета пропонованої роботи полягає в розв'язанні задачі Робена для гіперболічного рівняння четвертого порядку з умовами, що накладають зв'язок між певними похідними функції на межі. Для поставленої задачі використовуємо отриманий розв'язок задачі Коші для рівняння гіперболічного типу другого порядку. Розв'язок задачі Коші отримано за допомогою диференціальних операторів, що переводять довільні функції в регулярний розв'язок рівняння гіперболічного типу n -го порядку. Під час зведення задачі до розв'язаної задачі Коші для рівняння гіперболічного типу другого порядку й узгодження крайових умов цих задач виникає система рівнянь, розв'язок якої існує і є єдиним як розв'язок інтегрального рівняння Вольтерра, а також як розв'язок неоднорідного рівняння Бесселя. Розв'язком задачі Робена буде комбінація отриманих розв'язків відповідних задач Коші для рівнянь другого порядку.

Ключові слова: диференціальний оператор, регулярні розв'язки, ітеровані рівняння гіперболічного типу, задача Робена.

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