

UDC 519.21

DOI: <https://doi.org/10.17721/1812-5409.2026/1.8>

Yana MOKANU, PhD Student, Assist.

ORCID ID: 0009-0006-0752-6584

e-mail: [yana.mokanu3075@gmail.com](mailto:yana.mokanu3075@gmail.com)

National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute", Kyiv, Ukraine

## ON ERGODIC PROPERTY OF SOME LÉVY-TYPE PROCESSES IN $\mathbb{R}^d$

*In this paper, we investigate the ergodicity in total variation of the process  $X$ , related to some integro-differential operator with unbounded coefficients, and describe the speed of convergence to the respective invariant measure. Roughly speaking, a Markov process is said to be ergodic if there exists an invariant probability measure to which the transition probability of the considered process converges. In this case, the speed of convergence is also called the ergodic rate, and it plays an important role in further investigations of asymptotic properties, in particular, in simulating the processes, calculating their functionals, and also in limit theorems.*

*In the proofs, we use the Lyapunov-type criterion, which resembles the well-known from the theory of ordinary differential equations Lyapunov's second method. Some examples are provided.*

**Key words:** ergodicity, Lévy-type process, Lyapunov criterion, Lyapunov function.

**AMS 2020 classification:** 60G51, 37A50, 60G17.

### Introduction

Consider an integro-differential operator

$$Lf(x) = \ell(x) \cdot \nabla f(x) + \int_{\mathbb{R}^d \setminus \{0\}} (f(x+u) - f(x) - \nabla f(x) \cdot u \mathbb{1}_{|u| < 1}) \nu(x, du), \quad (1)$$

defined on the set  $C_0^2(\mathbb{R}^d)$  of twice continuously differentiable functions with compact support, where  $\ell(\cdot) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ , and the Lévy-type kernel  $\nu(x, du)$  satisfies

$$\int_{\mathbb{R}^d \setminus \{0\}} (1 \wedge |u|^2) \nu(x, du) < \infty \quad \text{for any } x \in \mathbb{R}^d.$$

To simplify the notation, we assume that the kernel  $\nu(x, du)$  does not have an atom at 0 and write  $\int_{\mathbb{R}^d}$  instead of  $\int_{\mathbb{R}^d \setminus \{0\}}$ .

Under certain conditions (see Section 1 below) one can associate with  $(L, C_0^2(\mathbb{R}^d))$  a Markov process  $X$ , whose transition probability kernel  $P_t(x, dy)$  possesses certain regularity properties. The process  $X$  is often called a Lévy-type process, because locally its structure resembles the structure of a Lévy process, e.g. a process with independent stationary increments.

The aim of this paper is to derive sufficient conditions for ergodicity of the process  $X$  related to (1) and find the ergodic rates.

This work further develops our previous studies on the ergodicity of Lévy-type processes: we began our investigation in (Knopova, & Mokanu, 2024) with the one-dimensional case and bounded Lévy-type kernel; continued in (Knopova, & Mokanu, 2025), where a Lévy-driven stochastic differential equation (SDE) with unbounded coefficients was considered in dimension  $d$ .

The long-time behaviour of Markov processes has been extensively investigated. Skorokhod in (Skorokhod, 1987) discussed the long-time behaviour of general Markov processes and solutions to SDEs, as well as stabilisation properties of solutions to systems of SDEs. Masuda in (Masuda, 2004), (Masuda, 2007) investigated a Lévy-driven Ornstein-Uhlenbeck process, in particular, the existence of a smooth density, ergodicity, and convergence rates. Sufficient conditions for recurrence, positive recurrence, and exponential ergodicity of one-dimensional Lévy-type processes generated by the integro-differential operator were derived in (Wang, 2008). In (Wang, 2013) sufficient conditions for exponential and strong ergodicity of SDEs driven by symmetric  $\alpha$ -stable processes were obtained. Next, exponential ergodicity of solutions to certain SDEs (in particular, SDEs of McKean-Vlasov type), damping Hamiltonian dynamics with state-dependent and non-local collisions, and Lévy driven Langevin dynamics with singular potentials was studied in (Liang, Majka, & Wang, 2021), (Bao, & Wang, 2023) and (Bao, Fang, & Wang, 2024), respectively. Ergodicity of a solution to the jump-diffusion SDE was studied by Wee in (Wee, 1999). In (Fort, & Roberts, 2005) and (Douc, Fort, & Guillin, 2009) sufficient conditions for ergodicity were given in the form of so-called  $(f, r)$ -modulated moments. In (Kulik, & Scheutzow, 2018) the problem of the uniqueness of an invariant probability measure of a Markov process was investigated; here the authors also derived sufficient conditions for the weak convergence of the transition probability to the invariant measure. Kulik in (Kulik, 2018) discussed the asymptotic behaviour of a wide range of Markov processes from simple Markov chains with finite state spaces to more sophisticated models (including solutions to Lévy driven SDEs), in particular, the author examined different types of ergodic rates and their further application in limit theorems; the question of weak ergodicity was addressed as well. Sandrić in (Sandrić, 2013) gave sufficient conditions for recurrence, transience, and ergodicity of stable-like processes in terms of their coefficients. Further, sufficient conditions for recurrence, transience, and ergodicity of stable-like Markov chains under certain assumptions on their density functions and drift were derived in (Sandrić, 2014). In (Sandrić, 2016) the author discussed recurrence, transience, ergodicity, strong, (sub)-exponential ergodicity, and mixing properties of Lévy-type processes driven by the infinitesimal generator.

The structure of this paper is as follows. In Section 1, we recall the notion of ergodicity, discuss the assumptions, and formulate our results. Proofs are given in Section 2 and examples in Section 3.

### 1. Main definitions and results

A Markov process  $(X_t)_{t \geq 0}$  on  $\mathbb{R}^d$  with a transition probability kernel  $P_t(x, dy)$  is called (strongly) *ergodic* (see (Sandrić, 2016)) if there exists an invariant probability measure  $\pi(\cdot)$  such that

$$\lim_{t \rightarrow \infty} \|P_t(x, \cdot) - \pi(\cdot)\|_{TV} = 0$$

for any  $x \in \mathbb{R}^d$ . Here  $\|\cdot\|_{TV}$  is the total variation norm. Moreover, one can describe the speed of convergence to the invariant measure, if certain assumptions are satisfied. The process is called (non-uniformly)  $\psi$ -ergodic (see (Sandrić, 2016) for polynomial and exponential rates), if there exists a function  $\psi \geq 0$ ,  $\psi(t) \rightarrow 0$  as  $t \rightarrow \infty$ , such that

$$\|P_t(x, \cdot) - \pi(\cdot)\|_{TV} \leq C(x)\psi(t), \quad t \geq 1, \quad (2)$$

for some  $C: \mathbb{R}^d \rightarrow [0, \infty)$ .

There are several possible ways to show ergodicity of a Markov process (see (Kulik, 2018, § 3.5)). We use the skeleton chain approach, described in (Kulik, 2018), which allows us to reduce the continuous-time setting to the discrete one by constructing a skeleton chain  $X^h := \{X_{nh}, n \in \mathbb{Z}_+\}$  and apply ergodic theorems, developed specifically for Markov chains. Under certain irreducibility assumptions (e.g., Dobrushin condition), ergodicity of the skeleton chain implies ergodicity of the initial process with the same convergence rate (cf. (Kulik, 2018, § 3.2), (Sandrić, 2016)).

To see if the chain is ergodic, one can check the (local) Dobrushin (or other forms of irreducibility condition, suitable for a particular task (see the discussion in (Kulik, 2018, p. 64))) and the Lyapunov-type conditions ((Kulik, 2018, Th. 2.8.9, Th. 3.2.3)). Note that the irreducibility-type assumption is crucial for the described approach, but if it fails, it is still possible for the process to be ergodic only in a weaker sense (see the discussion in (Kulik, 2018, § 4)).

A Markov chain  $X$  with a transition kernel  $\mathbb{P}(x, dy)$  satisfies the *local Dobrushin* condition (cf. (Kulik, 2018, p. 42)) on a measurable set  $B \subset \mathbb{R}^d \times \mathbb{R}^d$  if

$$\sup_{x, y \in B} \|\mathbb{P}(x, \cdot) - \mathbb{P}(y, \cdot)\|_{TV} < 2. \quad (3)$$

In practice, it is sufficient and reasonable to check if there exists an absolutely continuous component of  $\mathbb{P}(x, dy)$ , such that  $x \mapsto \mathbb{P}(x, \cdot)$  is continuous in  $L_1$  (see (Kulik, 2018, Prop. 2.9.1)), instead of proving (3) directly. This, in turn, is the case if the initial Markov process possesses a strictly positive transition probability density  $p_t(x, \cdot)$ , such that  $x \mapsto p_t(x, \cdot)$  is continuous in  $x$  in  $L_1$  (see (Sandrić, 2016)). It means that the respective semigroup  $(P_t)_{t \geq 0}$ ,  $P_t f(x) := \mathbb{E}^x f(X_t)$  possesses the  $C_b$ -Feller property, that is, it preserves the space of bounded continuous functions (see (Böttcher, Schilling, & Wang, 2013)). The Dobrushin condition ensures that the process converges to the same invariant measure, regardless of its starting point.

The *Lyapunov-type* condition (see (Kulik, 2018, p. 52)) allows us to establish moment estimates, responsible for (non-uniform) ergodic rates. Suppose that there exist:

- a norm-like function  $V: \mathbb{R}^d \rightarrow [1, \infty)$  (i.e.  $V(x) \rightarrow \infty$  as  $|x| \rightarrow \infty$ ), bounded on a compact  $K$ ;
- a function  $f: [1, \infty) \rightarrow (0, \infty)$ , that admits a non-negative increasing and concave extension to  $[0, \infty)$ ;
- a constant  $C > 0$ ;

such that the following relations hold:

$$\mathbb{E}^x V(X_1) - V(x) \leq -f(V(x)) + C, \quad x \in \mathbb{R}^d, \quad (4)$$

$$f\left(1 + \inf_{x \notin K} V(x)\right) > 2C.$$

Inequality (4) is called the *Lyapunov inequality*, and such a function  $V$  is called the *Lyapunov function*.

Given that we are in the continuous-time settings, it is reasonable to consider the Lyapunov-type condition in terms of the process  $X$  itself, namely, in terms of its *full generator*  $\mathcal{L}$ :

$$\mathcal{L}V(x) \leq -f(V(x)) + C, \quad (5)$$

where  $f, V$  and  $C$  have the same meaning as above.

The full generator is taken into consideration because Lyapunov functions are often unbounded and the domain of  $L$  is too narrow to include them. Recall the “stochastic version” of a *full generator*, see (Böttcher, Schilling, & Wang, 2013, (1.50)). The full generator of a measurable contraction semigroup  $(P_t)_{t \geq 0}$  on the space  $C_\infty(\mathbb{R}^d)$  of continuous functions vanishing at infinity is

$$\hat{L} := \left\{ (f, g) \in (\mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}^d)) : \quad M_t^{[f, g]} := f(X_t) - f(x) = \int_0^t g(X_s) ds, \quad x \in \mathbb{R}^d, t \geq 0, \right. \\ \left. \text{is a local } \mathcal{F}_t^X := \sigma(X_s, s \leq t) \text{ martingale} \right\}.$$

Note that in general  $\hat{L}$  is not single-valued, see the discussion in (Böttcher, Schilling, & Wang, 2013, p. 36).

Since for  $f \in C_0^2(\mathbb{R}^d)$

$$P_t f(x) - f(x) = \int_0^t P_s Lf(x) ds \quad \forall t \geq 0, x \in \mathbb{R}^d, \quad (6)$$

see (Ethier, & Kurtz, 1986, Ch. 1, Prop. 1.5), by the strong Markov property and the Dynkin formula  $(f, Lf) \subset \hat{L}$  for any  $f \in C_0^2(\mathbb{R}^d)$ . Further, it is not difficult to check that  $Lf(x) < \infty$  for  $f \in \mathcal{D}_0$ , where

$$\mathcal{D}_0 := \left\{ f \in C^2(\mathbb{R}^d) : \left| \int_{|u| \geq 1} (f(x+u) - f(x)) \nu(x, du) \right| < \infty \quad \forall x \in \mathbb{R}^d \right\}.$$

Using the localization procedure, the Ito formula, and taking the expectation, we derive (6) for any  $f \in \mathcal{D}_0$  and, consequently, we get that  $(f, Lf) \in \hat{L}$ ,  $f \in \mathcal{D}_0$ . We denote this extension of  $L$  to  $\mathcal{D}_0$  by  $\mathcal{L}$ .

Summarising, condition (5) for the initial process implies (4) for its skeleton chain (see (Kulik, 2018, Th. 3.2.3)), which, along with the Dobrushin condition, ensures the ergodicity in total variation of this chain and provides the (non-uniform) ergodic rate (cf. (Kulik, 2018, Th. 2.8.6, Th. 2.8.8)). As it was mentioned above, the ergodicity of the chain implies the ergodicity of the initial process with the same ergodic rate.

The speed of convergence to  $\pi$  in (2) as  $t \rightarrow \infty$  is (cf. (Kulik, 2018, Th. 2.8.8, Th. 3.2.3))

$$\psi(t) := \left( \frac{1}{f(F^{-1}(\gamma t))} \right)^\delta, \quad F(t) := \int_1^t \frac{dw}{f(w)}, \quad t \in (1, \infty), \quad (7)$$

where  $\gamma, \delta \in (0, 1)$ . Moreover,

$$\int_{\mathbb{R}^d} f(V(x)) \pi(dx) < \infty.$$

Our standing assumptions in this paper would be

There exists a solution  $\{X_t, t > 0\}$  to the martingale problem for  $(L, C_0^2(\mathbb{R}^d))$ ; (A1)

The skeleton chain  $X^h := \{X_{nh}, n \in \mathbb{Z}_+\}$  satisfies the Dobrushin condition. (A2)

Condition (A1) assures the existence of the process associated with (1) (which we call the *Markovian selection* related to  $L$ ), but it looks rather abstract and is not easy to check. The sufficient conditions ensuring (A1) were obtained in (Kühn, 2020) and are the following:

The symbol  $q$  is locally bounded, continuous in  $x$ , satisfies  $q(x, 0) = 0$ ; (A1')

$\lim_{R \rightarrow \infty} \sup_{|x| \leq R} \sup_{|\xi| \leq R^{-1}} |q(x, \xi)| = 0$ . (A1'')

Recall that the symbol  $q(x, \xi)$  of the operator  $L$  is a function, such that for  $f$  from the Schwartz space  $S(\mathbb{R}^d)$

$$Lf(x) = - \int_{\mathbb{R}^d} e^{i\xi \cdot x} q(x, \xi) \hat{f}(\xi) d\xi,$$

where  $\hat{f}$  is the Fourier transform of  $f$ , and

$$q(x, \xi) = -i\ell(x) \cdot \xi + \int_{\mathbb{R}^d} \left( 1 - e^{i\xi \cdot u} + i\xi \cdot u \mathbb{1}_{|u| \leq 1} \right) \nu(x, du).$$

Under (A1') and (A1''), there exists a solution to the Martingale problem for  $(L, C_0^\infty(\mathbb{R}^d))$ , which is strongly Markov, see (Kühn, 2020, Th. 1.1, Cor. 4.1).

The uniqueness of the process in the case of unbounded coefficients is harder to prove. Up to our knowledge, at the moment there are no general results establishing the uniqueness of the solution to the Martingale problem for  $(L, C_0^\infty(\mathbb{R}^d))$ , if the kernel  $\nu(x, du)$  is unbounded, unless one can rewrite  $L$  (using the change of measure) in the form, usually used in Lévy-driven SDEs, and check the conditions for the existence and uniqueness of the (strong or weak) solution to the respective SDE. We do not pursue this topic in the current article and confine ourselves with finding the conditions that guarantee the ergodicity of *some* Markovian selection, related to  $(L, C_0^\infty(\mathbb{R}^d))$ .

It is a kind of general knowledge that ergodicity can come from the drift or diffusion parts of the generator. We investigate the conditions on the Lévy-type measure under which (5) is satisfied. Our interest lies both in the drift- and noise-induced ergodicity. Further, we will show that in the second case, when the ergodic behaviour comes from the kernel  $\nu(x, \cdot)$  itself, the support of  $\nu(x, \cdot)$  must depend on the direction of the vector  $x$ .

Denote by  $|u|$  the norm of a vector  $u \in \mathbb{R}^d$ ; by  $\mathcal{B}(r)$  and  $\mathcal{B}^c(r)$  the ball centred at 0 with radius  $r$  and its complement, respectively; by  $\mathcal{A}(r, R)$  the annulus with the inner radius  $r$  and the outer radius  $R$ .

As a Lyapunov function we take

$$V(x) = 1 + \phi^p(x), \quad p \in (0, 1), \quad x \in \mathbb{R}^d, \quad (8)$$

where  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}_+$ ,  $\phi \in C^2(\mathbb{R}^d)$ , such that  $\phi(x) = |x|$  for  $|x| > 1$  and  $\phi(x) \leq |x|$  for  $|x| \leq 1$ .

We also denote by  $e_x = \frac{x}{\|x\|}$  the unit vector of  $x \in \mathbb{R}^d$ , and by  $\gamma_{x,u}$  the cosine of the angle between unit vectors  $e_x, e_u$ :

$$\gamma_{x,u} := \cos \angle(e_x, e_u).$$

The aim of the article is to find conditions ensuring the Lyapunov-type inequality (5) for  $\mathcal{L}$ . Our strategy is to select its negative and positive components to compare their impact on the generator. For the sake of simplicity, we consider separately the “drift” and “kernel” parts:

$$\begin{aligned} \mathcal{L}V(x) &= \nabla V(x) \cdot \left( \ell(x) + \int_{\mathcal{A}(1,|x|)} u \nu(x, du) \right) + \\ &+ \int_{\mathbb{R}^d} (V(x+u) - V(x) - \nabla V(x) \cdot u \mathbb{1}_{|u| \leq |x|}) \nu(x, du) =: \\ &=: \mathcal{L}^{drift}V(x) + \mathcal{L}^{ker}V(x). \end{aligned}$$

Consider the “drift” part. For  $|x|$  big enough  $V(x) = 1 + |x|^p$ , so we can transform

$$\mathcal{L}^{drift}V(x) = p|x|^{p-1} \left( |\ell(x)| \gamma_{x,\ell(x)} + \int_{\mathcal{A}(1,|x|)} \gamma_{x,u} |u| \nu(x, du) \right).$$

Since our ultimate goal is to compare the speed of all generator components, we define the first “growth” function

$$\phi^{drift}(x) := |x|^{p-1} \max \left( |\ell(x)|, \int_{\mathcal{A}(1,|x|)} |u| \nu(x, du) \right).$$

It is not hard to see that there exists  $\zeta \in [-2, 2]$ , such that

$$\mathcal{L}^{drift}V(x) \leq -p\zeta \phi^{drift}(x). \quad (9)$$

Additionally, fix  $\lambda \in (0, 1)$  and split the “kernel” part  $\mathcal{L}^{ker}$  into the “ball” and “tail” sub-components, based on the jump size  $|u|$ :

$$\begin{aligned} \mathcal{L}^{ker}V(x) &= \int_{\mathcal{B}(|x|^\lambda)} (V(x+u) - V(x) - \nabla V(x) \cdot u) \nu(x, du) + \\ &+ \int_{\mathcal{B}^c(|x|^\lambda)} (V(x+u) - V(x) - \nabla V(x) \cdot u \mathbb{1}_{\mathcal{A}(|x|^\lambda, |x|)}) \nu(x, du) =: \\ &=: \mathcal{L}^{ball}V(x) + \mathcal{L}^{tail}V(x). \end{aligned} \quad (10)$$

Further, again according to the jump size  $|u|$ , split the “tail” part:

$$\begin{aligned} \mathcal{L}^{tail}V(x) &= \int_{\mathcal{A}(|x|^\lambda, |x|)} (V(x+u) - V(x) - \nabla V(x) \cdot u) \nu(x, du) + \\ &+ \int_{\mathcal{B}^c(|x|)} (V(x+u) - V(x)) \nu(x, du) =: \\ &=: \mathcal{L}^{big}V(x) + \mathcal{L}^{large}V(x). \end{aligned}$$

Now we are ready to introduce the rest of the “growth” functions:

$$\phi_+^{ball}(x) := |x|^{p-2} \int_{\mathcal{B}(|x|^\lambda)} |u|^2 \nu(x, du), \quad \phi_-^{ball}(x) := |x|^{p-2} \int_{\mathcal{B}(|x|^\lambda)} |u|^2 \gamma_{x,u}^2 \nu(x, du),$$

and

$$\phi^{big}(x) := \frac{p}{2} |x|^{p-1} \int_{\mathcal{A}(|x|^\lambda, |x|)} |u| \nu(x, du), \quad \phi^{large}(x) := \int_{\mathcal{B}^c(|x|)} |u|^p \nu(x, du).$$

Therefore, we arrive at the following result.

**Lemma 1.** For  $V$  from (8) and  $\zeta \in [-2, 2]$ :

$$\mathcal{L}V(x) \leq -p\zeta \phi^{drift}(x) + \frac{p}{4} (1 + o(1)) (\phi_+^{ball}(x) + (p-2) \phi_-^{ball}(x)) + \phi^{big}(x) + \phi^{large}(x), \quad |x| \rightarrow \infty. \quad (11)$$

Inequality (11) gives us a first vision on how the final condition may look. Indeed, consider the right-hand side of (11). The last two terms (which relate to “tail”) are positive by construction. Consequently, the negativity of  $\mathcal{L}V$  can come from either the “drift” or “ball” components, or a combination thereof. According to this, we introduce several constants that illustrate the interconnection of the corresponding “growth” functions.

First, it can be seen from (11) that  $\phi_+^{ball}$  and  $\phi_-^{ball}$  correspond to the positive and negative parts of “ball” respectively thus, the sign of  $\mathcal{L}^{ball}V$  is fully determined by their relation. Define

$$C^{ball} := \limsup_{|x| \rightarrow \infty} \frac{\phi_+^{ball}(x)}{\phi_-^{ball}(x)}.$$

By definition  $C^{ball} \geq 1$  and it has to be less than  $2 - p$  for  $\mathcal{L}^{ball}V$  to be negative. In other words, it means that the measure  $\nu(x, \cdot)$  has to be concentrated along the direction, collinear to  $x$ .

To compare “drift” with “tail” we introduce

$$C^{big} := \limsup_{|x| \rightarrow \infty} \frac{\phi^{big}(x)}{\phi^{drift}(x)} \quad \text{and} \quad C^{large} := \limsup_{|x| \rightarrow \infty} \frac{\phi^{large}(x)}{\phi^{drift}(x)}.$$

In the same vein, the constant

$$C^{tail} := \limsup_{|x| \rightarrow \infty} \frac{\max(\phi^{big}(x), \phi^{large}(x))}{\phi_-^{ball}(x)}. \quad (12)$$

shows the relation between “ball” and “tail”.

In both cases, finite constants ensure that “tail” does not spoil the negativity of  $\mathcal{L}V$ , and under additional assumptions, one can get either “drift”- or “ball”-induced ergodicity (respectively).

**Remark 1.** In (12) we do not consider the constant, related to  $\phi_+^{ball}$ , because in the case of “ball”-induced ergodicity, the strictly negative component of “ball” (that is  $\phi_-^{ball}$ ) has to dominate.

Finally, to compare “drift” and “ball”, we define

$$\tilde{C}_+^{drift} := \limsup_{|x| \rightarrow \infty} \frac{\phi_+^{ball}(x)}{\phi^{drift}(x)} \quad \text{and} \quad \tilde{C}_-^{drift} := \liminf_{|x| \rightarrow \infty} \frac{\phi_-^{ball}(x)}{\phi^{drift}(x)}.$$

Note that the functions  $\phi^{drift}$  and  $\phi_-^{ball}$  are not necessarily monotone; thus, in order to formulate our main result in a more “user-friendly” manner, we assume that they can be bounded from below by some power-type functions. Namely, suppose that for  $|x|$  big enough there exist some strictly positive  $K^{drift}$ ,  $K^{ball}$ , and  $\gamma^{drift}$ ,  $\gamma^{ball}$ , such that the following inequalities hold:

$$\phi^{drift}(x) \geq K^{drift}|x|^{\gamma^{drift}}, \quad \phi_-^{ball}(x) \geq K^{ball}|x|^{\gamma^{ball}}. \quad (13)$$

Now we are ready to formulate the main results of the paper.

**Theorem 1. Case 1. (“Drift-induced” ergodicity).** Suppose that (9) holds with  $\zeta \in (0, 2]$ , and  $\tilde{C}_+^{drift}$ ,  $\tilde{C}_-^{drift}$ ,  $C^{big}$ ,  $C^{large}$  are finite. Then (5) holds with  $f(u) = Ku^{\gamma^{drift}/p}$ ,  $K > 0$ , if

$$-p\zeta + \frac{p}{4} \left( \tilde{C}_+^{drift} + (p-2)\tilde{C}_-^{drift} \right) + C^{big} + C^{large} < 0. \quad (14)$$

**Case 2. (“Ball-induced” ergodicity).** Suppose that  $C^{ball}$ ,  $C^{tail} < \infty$ . Then (5) holds with  $f(u) = Ku^{\gamma^{ball}/p}$ ,  $K > 0$ , if either

a)  $\tilde{C}_-^{drift} = \infty$ ,  $\zeta \in [-2, 2]$  and

$$\frac{p}{4}(C^{ball} + p - 2) + 2C^{tail} < 0. \quad (15)$$

or

b)  $\tilde{C}_-^{drift} \in (0, \infty)$ ,  $\zeta \in [-2, 0]$  and

$$-\frac{p\zeta}{\tilde{C}_-^{drift}} + \frac{p}{4}(C^{ball} + p - 2) + 2C^{tail} < 0. \quad (16)$$

**Remark 2.** In Case 2 we have purely “ball”-induced ergodicity: in Case 2a) “drift” does not have any impact compared to “ball” and in Case 2b) “drift” is positive and thus works even against. In Case 1 ergodicity comes from “drift” only, if  $\tilde{C}_-^{drift} = 0$ ; otherwise both components contribute.

## 2. Proofs

**Proof of Lemma 1.** We already have the estimate of  $\mathcal{L}^{drift}V$  (see (9)). Consider now  $\mathcal{L}^{ball}V$ . Since  $|u| \in \mathcal{B}(|x|^\lambda)$ , then for  $|x| \gg 1$ :

$$|x + u| \geq |x| \cdot |1 - |x|^{\lambda-1}| > 1,$$

and  $V(x+u) = 1 + |x+u|^p$ . Thus, we can apply Taylor's formula to the integrand function of the first term of (10) and get

$$V(x+u) - V(x) - \nabla V(x) \cdot u = \frac{1}{2} \int_0^1 (1-\theta) u' D^2 V(x+\theta u) u d\theta,$$

where

$$D^2 V(x) = \{\partial_{ij}^2 V(x)\}_{i,j=1}^d = p|x|^{p-2} \begin{pmatrix} 1 + (p-2) \frac{x_1^2}{|x|^2} & (p-2) \frac{x_1 x_2}{|x|^2} & \cdots & (p-2) \frac{x_1 x_d}{|x|^2} \\ (p-2) \frac{x_1 x_2}{|x|^2} & 1 + (p-2) \frac{x_2^2}{|x|^2} & \cdots & (p-2) \frac{x_2 x_d}{|x|^2} \\ \cdots & \cdots & \ddots & \cdots \\ (p-2) \frac{x_1 x_d}{|x|^2} & (p-2) \frac{x_2 x_d}{|x|^2} & \cdots & 1 + (p-2) \frac{x_d^2}{|x|^2} \end{pmatrix}.$$

Then

$$\begin{aligned} \mathcal{L}^{ball} V(x) &= \frac{1}{2} \int_{\mathbb{B}(|x|^\lambda)} \int_0^1 (1-\theta) u' \cdot D^2 V(x+\theta u) \cdot u \nu(x, du) d\theta = \\ &= \frac{p}{2} \int_{\mathbb{B}(|x|^\lambda)} \int_0^1 (1-\theta) |x+\theta u|^{p-2} \left( \sum_{i=1}^d \left( 1 + (p-2) \frac{(x_i + \theta u_i)^2}{|x+\theta u|^2} \right) u_i^2 + \right. \\ &\quad \left. + (p-2) \sum_{i,j=1, i \neq j}^d u_i u_j \frac{(x_i + \theta u_i)(x_j + \theta u_j)}{|x+\theta u|^2} \right) d\theta \nu(x, du). \end{aligned}$$

Regroup the sums in the inner integral and apply the sum of squares formula to  $u_i(x_i + \theta u_i)$ :

$$\begin{aligned} \mathcal{L}^{ball} V(x) &= \frac{p}{2} \int_{\mathbb{B}(|x|^\lambda)} \int_0^1 (1-\theta) |x+\theta u|^{p-2} \left( \sum_{i=1}^d u_i^2 + \frac{p-2}{|x+\theta u|^2} \left( \sum_{i=1}^d u_i(x_i + \theta u_i) \right)^2 \right) d\theta \nu(x, du) = \\ &= \frac{p}{2} \int_{\mathbb{B}(|x|^\lambda)} \int_0^1 (1-\theta) |x+\theta u|^{p-2} |u|^2 \left( 1 + \underbrace{(p-2) (e_u \cdot e_{x+\theta u})^2}_{\text{negative}} \right) d\theta \nu(x, du). \end{aligned}$$

Consider the squared scalar product

$$\begin{aligned} (e_u \cdot e_{x+\theta u})^2 &= \left( \gamma_{x,u} \frac{|x|}{|x+\theta u|} + \frac{\theta |u|}{|x+\theta u|} \right)^2 \geq \gamma_{x,u}^2 \left( \frac{|x|}{|x|+|u|} \right)^2 - \frac{2\theta |x| \cdot |u|}{(|x|+|u|)^2} \geq \\ &\geq \frac{\gamma_{x,u}^2}{(1+|x|^{\lambda-1})^2} - \frac{2|x|^\lambda}{|x|(1-|x|^{\lambda-1})^2}. \end{aligned}$$

Estimating the inner integral of  $\mathcal{L}^{ball} V$  depending upon its sign (again by triangle inequalities), we get

$$\frac{1}{2} (|x| \cdot |1 + |x|^{\lambda-1}|)^{p-2} \leq \int_0^1 (1-\theta) |x+\theta u|^{p-2} d\theta \leq \frac{1}{2} (|x| \cdot |1 - |x|^{\lambda-1}|)^{p-2}.$$

Summarising, we derive the final estimate of the "ball" part:

$$\begin{aligned} \mathcal{L}^{ball} V(x) &\leq \frac{p}{4} |x|^{p-2} \int_{\mathbb{B}(|x|^\lambda)} |u|^2 (C_1(x) + (p-2) C_2(x) \gamma_{x,u}^2) \nu(x, du) = \\ &= \frac{p}{4} C_1(x) \phi_+^{ball}(x) + \frac{p(p-2)}{4} C_2(x) \phi_-^{ball}(x), \end{aligned}$$

where

$$C_1(x) = \frac{1}{(1-|x|^{\lambda-1})^{2-p}} + \frac{2(2-p)|x|^{\lambda-1}}{(1+|x|^{\lambda-1})^{2-p}(1-|x|^{\lambda-1})^2}, \quad C_2(x) = \frac{1}{(1+|x|^{\lambda-1})^{4-p}}.$$

Both  $C_1(x)$  and  $C_2(x)$  tend to 1 as  $|x|$  tends to infinity; therefore for  $|x|$  large enough:

$$\mathcal{L}^{ball} V(x) \leq \frac{p}{4} (1+o(1)) (\phi_+^{ball}(x) + (p-2) \phi_-^{ball}(x)). \quad (17)$$

Next, consider the "tail" part, beginning with the big jumps. Since by definition  $V(x)$  is either  $1 + |x|^p$  or  $1 + \phi^p(x)$

(depending on the value of  $|x|$ ), and  $\phi(x) \leq |x|$ , then  $V(x+u) \leq 1 + |x+u|^p$ , and

$$\begin{aligned} \mathcal{L}^{big} V(x) &\leq \int_{\mathcal{A}(|x|^\lambda, |x|)} (|x+u|^p - |x|^p - p|x|^{p-1} \cdot |u|\gamma_{x,u}) \nu(x, du) = \\ &= |x|^p \int_{\mathcal{A}(|x|^\lambda, |x|)} \left( \left( 1 + 2\frac{|u|}{|x|} \gamma_{x,u} + \left( \frac{|u|}{|x|} \right)^2 \right)^{\frac{p}{2}} - 1 \right) \nu(x, du) - \\ &- p|x|^{p-1} \int_{\mathcal{A}(|x|^\lambda, |x|)} |u|\gamma_{x,u} \nu(x, du) =: J_1(x) + J_2(x). \end{aligned}$$

Now using the fact that  $u \in \mathcal{A}(|x|^\lambda, |x|)$ , we derive

$$\begin{aligned} J_1(x) &\leq |x|^p \int_{\mathcal{A}(|x|^\lambda, |x|)} \left( \left( 1 + (1 + 2\gamma_{x,u}) \frac{|u|}{|x|} \right)^{\frac{p}{2}} - 1 \right) \nu(x, du) \leq \\ &\leq \frac{p}{2} |x|^{p-1} \int_{\mathcal{A}(|x|^\lambda, |x|)} (1 + 2\gamma_{x,u}) |u| \nu(x, du). \end{aligned}$$

Summing  $J_1(x)$  and  $J_2(x)$ , we obtain the final estimate of the big jumps part:

$$\mathcal{L}^{big} V(x) \leq \frac{p}{2} |x|^{p-1} \int_{\mathcal{A}(|x|^\lambda, |x|)} |u| \nu(x, du) = \phi^{big}(x). \quad (18)$$

In the case of large jumps we again use that by definition  $V(x+u) \leq 1 + |x+u|^p$ . Then

$$\mathcal{L}^{large} V(x) \leq \int_{\mathcal{B}^c(|x|)} (|x+u|^p - |x|^p) \nu(x, du) \leq \int_{\mathcal{B}^c(|x|)} |u|^p \nu(x, du) = \phi^{large}(x). \quad (19)$$

Together estimates (9), (17), (18) and (19) yield the statement of the lemma, namely,

$$\mathcal{L} V(x) \leq -p\zeta \phi^{drift}(x) + \frac{p}{4} (1 + o(1)) (\phi_+^{ball}(x) + (p-2)\phi_-^{ball}(x)) + \phi^{big}(x) + \phi^{large}(x), \quad |x| \rightarrow \infty.$$

□

Now, using the results of the proven lemma, we can easily prove Theorem 1.

**Proof of Theorem 1. Case 1. "Drift-induced" ergodicity.** Suppose that (9) holds with  $\zeta \in (0, 2]$ . Dividing both sides of (11) by  $\phi^{drift}$  and taking  $\limsup$  as  $|x| \rightarrow \infty$ , we get

$$\limsup_{|x| \rightarrow \infty} \frac{\mathcal{L} V(x)}{\phi^{drift}(x)} \leq -p\zeta + \frac{p}{4} \left( \tilde{C}_+^{drift} + (p-2)\tilde{C}_-^{drift} \right) + C^{big} + C^{large}. \quad (20)$$

Then it is clear that the Lyapunov-type condition (5) is satisfied if  $\tilde{C}_+^{drift}$ ,  $\tilde{C}_-^{drift}$ ,  $C^{big}$ ,  $C^{large}$  are finite, and moreover the right-hand side of (20) is strictly negative.

**Case 2. "Ball-induced" ergodicity.**

a) Assume that  $\tilde{C}_-^{drift} = \infty$  (that is, "ball" is way faster than "drift") and  $\zeta \in [-2, 2]$ . Dividing this time both sides of (11) by  $\phi_-^{ball}$  and taking  $\limsup$  as  $|x| \rightarrow \infty$ , we get

$$\limsup_{|x| \rightarrow \infty} \frac{\mathcal{L} V(x)}{\phi_-^{ball}(x)} \leq \frac{p}{4} (C^{ball} + p-2) + 2C^{tail}. \quad (21)$$

If  $C^{ball}, C^{tail} < \infty$ , and the right-hand side of (21) is strictly negative, condition (5) is satisfied.

b) Assume  $\tilde{C}_-^{drift} \in (0, \infty)$  and  $\zeta \in [-2, 0]$ . Following the same approach as above we get

$$\limsup_{|x| \rightarrow \infty} \frac{\mathcal{L} V(x)}{\phi_-^{ball}(x)} \leq -\frac{p\zeta}{\tilde{C}_-^{drift}} + \frac{p}{4} (C^{ball} + p-2) + 2C^{tail}. \quad (22)$$

If  $C^{ball}, C^{tail} < \infty$ , and the right-hand side of (22) is strictly negative, condition (5) holds.

In each case, the form of  $f$  from (5) (and thus of the ergodic rates  $\psi$  from (7)) is determined by (13).

□

### 3. Examples

**Example.** Consider a one-dimensional symmetric bounded  $\nu(\cdot, \cdot)$  and suppose that the drift part  $\ell(x)$  is absent. Then  $\phi^{drift}$ ,  $\phi^{big}$ , and  $\phi^{large}$  are all equal to zero for all  $x \in \mathbb{R}$ , and we are in the setting of Case 2a) with  $C^{tail} = 0$ . Moreover, given that we are in the one-dimensional setting,  $|\gamma_{x,u}| = 1$ , thus,  $\phi_+^{ball} = \phi_-^{ball}$  and  $C^{ball} = 1$ . Then (15) is always satisfied, yielding the Lyapunov-type condition (5).

**Example.** Consider now an  $\alpha$ -stable Lévy-type measure  $\nu(x, du) = |x|^\beta |u|^{-d-\alpha} du$ ,  $x \in \mathbb{R}^d$ , with  $\beta \in (1-p, 1)$  and  $\alpha \in (0, 2)$ . Take  $\ell(x) = |x|^\eta$ ,  $\eta \in (0, \beta)$ , and assume that  $\zeta \in [1/2, 1]$ .

First, estimate

$$\phi_-^{ball}(x) \leq \phi_+^{ball}(x) = \frac{S(d-1)}{2-\alpha} |x|^{p-2+\beta+\lambda(2-\alpha)},$$

where  $S(d-1)$  is the surface area of the unit sphere of dimension  $d-1$ .

Next, to calculate the rest of the “growth” functions we define

$$\phi_\tau^{ann}(x) := |x|^{p-1+\beta} \int_{\mathcal{A}(|x|^\tau, |x|)} |u|^{1-d-\alpha} du, \quad \tau \in \{0, \lambda\}.$$

Simple calculations lead us to

$$\phi_\tau^{ann}(x) \asymp S(d-1) \begin{cases} (\alpha-1)^{-1} |x|^{p-1+\beta+(1-\alpha)\tau}, & \alpha > 1, \\ (1-\alpha)^{-1} |x|^{p+\beta-\alpha}, & \alpha < 1, \\ (1-\tau) |x|^{p-1+\beta} \ln |x|, & \alpha = 1. \end{cases}$$

Then (remember that  $\eta \in (0, \beta)$ )

$$\phi^{drift}(x) = \phi_0^{ann}(x), \quad \phi^{big}(x) = \frac{p}{2} \phi_\lambda^{ann}(x).$$

Finally,

$$\phi^{large}(x) = \frac{S(d-1)}{\alpha-p} |x|^{p+\beta-\alpha}.$$

We can easily see that  $\tilde{C}_+^{drift}$  (hence and  $\tilde{C}_-^{drift}$ ) are equal to zero for every  $\alpha \in (0, 2)$ . Furthermore, given that  $\zeta \in [1/2, 1]$ , inequality (14) is automatically satisfied for  $\alpha \geq 1$ , thus, Lyapunov-type condition (5) holds with  $f(u) = Cu^{\frac{p-1+\beta}{p}}$ ,  $C \leq p\zeta$ , for  $\alpha > 1$ , and with  $f(u) = Cu^{\frac{p-1+\beta}{p}} \ln u$ ,  $C \leq \frac{p}{2}(2\zeta-1+\lambda)$ , for  $\alpha = 1$ . For  $\alpha < 1$ , (5) holds with  $f(u) = Cu^{\frac{p+\beta-\alpha}{p}}$ ,  $C \leq p(\zeta - \frac{1}{2}) - \frac{1-\alpha}{\alpha-p}$ , if  $p$  satisfies the inequality

$$p^2(1-2\zeta) - p\alpha(1-2\zeta) - 2(1-\alpha) > 0.$$

In each case,  $C$  was derived directly from (14).

### Discussion and conclusions

In this paper, we investigated the long-time behaviour of the process  $X$ , related to some integro-differential operator with unbounded coefficients. Using the Lyapunov-type criterion, we found the conditions on the Lévy-type measure of the process  $X$  ensuring its ergodicity in total variation norm; in particular, we showed that in the case of noise-induced ergodicity the Lévy-type measure must be concentrated along the direction, collinear to the spatial coordinate  $x$ . The conditions were given in terms of the coefficients of the process’s generator, making the results easily applicable in practice.

**Acknowledgments.** The author expresses deepest gratitude to her supervisor, Dr. Victoria Knopova, for her mentorship and constant encouragement. The author sincerely thanks Prof. Dr. René Schilling for the hospitality and valuable, insightful comments and hints. The author is also grateful to the anonymous reviewers for their comments and constructive suggestions, which helped improve the quality and clarity of this manuscript.

**Sources of funding.** This study did not receive any grant from a funding institution in the public, commercial, or non-commercial sectors.

### References

- Bao, J., Fang, R., & Wang, J. (2024). Exponential ergodicity of Lévy driven Langevin dynamics with singular potentials. *Stochastic Processes and their Applications*, 172, 104341. <https://doi.org/10.1016/j.spa.2024.104341>
- Bao, J., & Wang, J. (2023). Exponential ergodicity for damping Hamiltonian dynamics with state-dependent and non-local collisions. *Bernoulli*, 29(3), 2442–2465. <https://doi.org/10.3150/22-BEJ1548>
- Böttcher, B., Schilling, R., & Wang, J. (2013). *Lévy matters. III*. Springer, Cham. <https://doi.org/10.1007/978-3-319-02684-8>
- Douc, R., Fort, G., & Guillin, A. (2009). Subgeometric rates of convergence of  $f$ -ergodic strong Markov processes. *Stochastic Processes and their Applications*, 119(3), 897–923. <https://doi.org/10.1016/j.spa.2008.03.007>
- Ethier, S. N., & Kurtz, T. G. (1986). *Markov processes. Characterization and convergence*. John Wiley & Sons. <https://doi.org/10.1002/9780470316658>
- Fort, G., & Roberts, G. O. (2005). Subgeometric ergodicity of strong Markov processes. *The Annals of Applied Probability*, 15(2), 1565–1589. <https://doi.org/10.1214/105051605000000115>
- Knopova, V., & Mokanu, Y. (2024). On ergodic properties of some Lévy-type processes. *Journal of Theoretical Probability*, 37(1), 582–602. <https://doi.org/10.1007/s10959-023-01252-x>
- Knopova, V., & Mokanu, Y. (2025). On ergodic property of the solution to a Lévy-driven SDE. *Cybernetics and Systems Analysis*, 61(5), 98–109. <https://doi.org/10.1007/s10559-025-00830-7>

- Kühn, F. (2020). Existence of (Markovian) solutions to martingale problems associated with Lévy-type operators. *Electronic Journal of Probability*, 25, 1–26. <https://doi.org/10.1214/20-ejp424>
- Kulik, A. (2018). *Ergodic behavior of Markov processes: With applications to limit theorems*. De Gruyter. <https://doi.org/10.1515/9783110458930>
- Kulik, A., & Scheutzow, M. (2018). Generalized couplings and convergence of transition probabilities. *Probability Theory and Related Fields*, 171, 333–376. <https://doi.org/10.1007/s00440-017-0779-8>
- Liang, M., Majka, M., & Wang, J. (2021). Exponential ergodicity for SDEs and McKean–Vlasov processes with Lévy noise. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 57(3), 1665–1701. <https://doi.org/10.1214/20-AIHP1123>
- Masuda, H. (2004). On multidimensional Ornstein–Uhlenbeck processes driven by a general Lévy process. *Bernoulli*, 10(1), 97–120. <https://doi.org/10.3150/bj/1077544605>
- Masuda, H. (2007). Ergodicity and exponential  $\beta$ -mixing bounds for multidimensional diffusions with jumps. *Stochastic Processes and their Applications*, 117(1), 35–56. <https://doi.org/10.1016/j.spa.2006.04.010>
- Sandrić, N. (2013). Long-time behavior of stable-like processes. *Stochastic Processes and their Applications*, 123(4), 1276–1300. <https://doi.org/10.1016/j.spa.2012.12.004>
- Sandrić, N. (2014). Ergodic property of stable-like Markov chains. *Journal of Theoretical Probability*, 29(2), 459–490. <https://doi.org/10.1007/s10959-014-0586-4>
- Sandrić, N. (2016). Ergodicity of Lévy-type processes. *ESAIM. Probability and Statistics*, 20, 154–177. <https://doi.org/10.1051/ps/2016009>
- Skorokhod, A. (1987). *Asymptotic methods of the theory of stochastic differential equations*. Naukova Dumka [in Russian]. [Скороход, А. (1987). *Асимптотические методы теории стохастических дифференциальных уравнений*. Наукова думка].
- Wang, J. (2008). Criteria for ergodicity of Lévy type operators in dimension one. *Stochastic Processes and their Applications*, 118(10), 1909–1928. <https://doi.org/10.1016/j.spa.2007.11.003>
- Wang, J. (2013). Exponential ergodicity and strong ergodicity for SDEs driven by symmetric  $\alpha$ -stable processes. *Applied Mathematics Letters*, 26, 654–658. <https://doi.org/10.1016/j.aml.2013.01.004>
- Wee, I.-S. (1999). Stability for multidimensional jump-diffusion processes. *Stochastic Processes and their Applications*, 80(2), 193–209. [https://doi.org/10.1016/S0304-4149\(98\)00078-7](https://doi.org/10.1016/S0304-4149(98)00078-7)

Отримано редакцією журналу / Received: 13.09.25

Прорецензовано / Revised: 12.03.26

Схвалено до друку / Accepted: 16.04.26

Опубліковано / Published: 05.06.26

Яна МОКАНУ, асп., асист.

ORCID ID: 0009-0006-0752-6584

e-mail: yana.mokanu3075@gmail.com

Національний технічний університет України “Київський політехнічний інститут імені Ігоря Сікорського”, Київ, Україна

## ЕРГОДИЧНІСТЬ ДЕЯКИХ ПРОЦЕСІВ ТИПУ ЛЕВІ В $\mathbb{R}^d$

Досліджено ергодичність у повній варіації процесу  $X$ , пов'язаного з деяким інтегро-диференціальним оператором з необмеженими коефіцієнтами, та описано швидкість збіжності до відповідної інваріантної міри. Іншими словами, процес Маркова є ергодичним, якщо існує інваріантна ймовірнісна міра, до якої збігається ядро ймовірності переходу відповідного процесу. У такому разі швидкість збіжності називається коефіцієнтом ергодичності; вона відіграє важливу роль у подальшому вивченні асимптотичних властивостей, зокрема в моделюванні процесів, обчисленні їхніх функціоналів, а також у граничних теоремах.

Для доведення застосовано критерій типу Ляпунова, що нагадує відомий з теорії звичайних диференціальних рівнянь другий метод Ляпунова. Теоретичні результати проілюстровано прикладами.

**Ключові слова:** ергодичність, процес типу Леві, критерій Ляпунова, функція Ляпунова.

Автор заявляє про відсутність конфлікту інтересів. Спонсори не брали участі в розробленні дослідження; у зборі, аналізі чи інтерпретації даних; у написанні рукопису; в рішенні про публікацію результатів.

The author declares no conflicts of interest. The funders had no role in the design of the study; in the collection, analyses or interpretation of data; in the writing of the manuscript; in the decision to publish the results.