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WEAK CONVERGENCE ANALYSIS OF DYNAMIC CUTTING FOR LÉVY-TYPE PROCESSES

We derive weak error bounds for Euler – Maruyama schemes of the one-dimensional Lévy-driven SDE, where small jumps are truncated in a time-dependent way using a dynamic cutting technique. Large jumps are simulated exactly, while small jumps are either (i) omitted or (ii) replaced by a Gaussian term with matching variance. Under standard Lipschitz-growth and smoothness conditions on the coefficients and Lévy measure, we prove that the weak error of both schemes is of order $O(n^{-1})$. This rate is achieved by choosing the scaling hyperparameter as $h = n^{-\alpha/(\varepsilon(2-\alpha))}$ for (i) and $h = n^{-\alpha/(\varepsilon(3-\alpha))}$ for (ii) respectively.

Key words: dynamic cutting, Lévy-driven SDEs, weak convergence, Euler – Maruyama scheme.

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Introduction

We consider a Lévy-driven SDE of the type

$$X_t = X_0 + \int_0^t a(s, X_s) ds + \int_0^t b(s, X_s) dB_s + \int_0^t \int_{-\infty}^{\infty} c(s, X_{s-}, z) \tilde{N}(ds, dz), \quad (1)$$

where $\tilde{N}(ds, dz)$ is the compensated Poisson random measure corresponding to a Lévy process Z_t , and $a(s, x)$, $b(s, x)$, $c(s, x, z)$ are some measurable functions, conditions on which we specify in Section 2.

The analysis of weak approximation errors for Lévy-driven SDEs began with the work of (Protter, & Talay, 1997), who established that the error of the Euler scheme can be expanded in powers of $1/n$. Building on this, (Jacod, 2004) established functional limit theorems for the normalized error process.

A main practical obstacle is the infinite activity of many Lévy processes, making the exact simulation of small jumps impossible. To tackle this (Asmussen, & Rosiński, 2001) introduced a truncation method where jumps below a fixed threshold ε are removed and their effect is approximated by a Gaussian term with suitable variance. Recently, (Bossy, & Maurer, 2025) revisited the fixed-cutoff for the time-inhomogeneous SDEs, demonstrating that the weak convergence rate of $1/n + \varepsilon^{3-\beta}$ can be achieved by choosing threshold ε optimally.

Classical approaches rely on a *fixed* truncation level. In contrast, time-dependent threshold for simulating Lévy processes was introduced by (Ivanenko, Knopova, & Platonov, 2025). Later, the same *dynamic cutting* approach was applied to analyze strong error bounds of the time-inhomogeneous SDEs (Knopova, & Platonov, 2025). In this paper, we transfer this *dynamic cutting* framework to weak error analysis.

1. Dynamic cutting setup

We assume that the driving jump process Z admits the Lévy-Itô representation

$$Z_t = \int_0^t \int_{\mathbb{R}} z (N(ds, dz) - \nu(dz)ds) = \int_0^t \int_{\mathbb{R}} z \tilde{N}(ds, dz), \quad (2)$$

where $N((0, t], B)$ counts the jumps of Z that fall in B , $\tilde{N}(ds, dz) = N(ds, dz) - \nu(dz)ds$ denotes the compensated Poisson random measure, and ν is the associated Lévy measure.

Let $\nu^+ := \nu|_{(0, +\infty)}$ and $\nu^- := \nu|_{(-\infty, 0)}$. Denote for $r > 0$

$$N^+(r) := \nu^+(r, +\infty), \quad N^-(r) := \nu^-(-\infty, -r),$$

and their generalized inverses at t^{-1} for $t > 0$ by

$$\tau^\pm(t) := \sup\{r \geq 0 : N^\pm(r) \geq t^{-1}\}, \quad (3)$$

$$\tau(t) := \max(\tau^+(t), \tau^-(t)). \quad (4)$$

For $s > 0$, define

$$B(s, h, \varepsilon) := \{z : -\tau^-((sh)^\varepsilon) < z < \tau^+((sh)^\varepsilon)\}.$$

Here, $h \in (0, 1)$ and $\varepsilon \in (0, 1)$ are the dynamic cutting parameters introduced in (Ivanenko, Knopova, & Platonov, 2025). They are responsible for controlling the shape and scaling of the time-dependent threshold.

Consider two types of Euler-Maruyama schemes:

$$X_{t_{i+1}}^{1,\tau} := X_{t_i}^{1,\tau} + a(t_i, X_{t_i}^{1,\tau})\Delta_i + b(t_i, X_{t_i}^{1,\tau})\sqrt{\Delta_i}\xi_i + C_{t_i}(X_{t_i}^{1,\tau}), \quad (5)$$

$$X_{t_{i+1}}^{2,\tau} := X_{t_i}^{2,\tau} + a(t_i, X_{t_i}^{2,\tau})\Delta_i + b(t_i, X_{t_i}^{2,\tau})\sqrt{\Delta_i}\xi_i + C_{t_i}(X_{t_i}^{2,\tau}) + \sigma_i(X_{t_i}^{2,\tau})\zeta_i, \quad (6)$$

where:

- ξ_i, ζ_i – sequences of i.i.d. standard Gaussian random variables;
- $C_{t_i}(x)$ is the compensated frozen large-jump increment

$$C_{t_i}(x) := \int_{t_i}^{t_{i+1}} \int_{\mathbb{R} \setminus B(t_i, h, \varepsilon)} c(t_i, x, z) \tilde{N}(ds, dz).$$

- $\sigma_i^2(x)$ is the variance of the Gaussian correction and uses the frozen left-endpoint value

$$\sigma_i^2(x) := \Delta_i \int_{B(t_i, h, \varepsilon)} c^2(t_i, x, z) \nu(dz).$$

For simplicity, we write X_t^τ to denote $X_t^{1,\tau}$ or $X_t^{2,\tau}$ when the distinction between the schemes is not relevant.

Let $0 = t_0 < t_1 < \dots < t_n = T$ be the grid formed by combining equally spaced points and large jumps (positive and negative). This construction guarantees that large jumps are included as grid points. In the weak-error proof below, we use the standard frozen-coefficient continuous Euler interpolation associated with these grid values and, for notational simplicity, denote this interpolation again by X_t^τ .

In addition, let us define the lower Pruitt functions as

$$\psi^{L,\pm}(\xi) := \int_{\{u \neq 0, |u\xi| \leq 1\}} |u\xi|^2 \nu^\pm(du), \quad \psi^L := \psi^{L,+} + \psi^{L,-}. \quad (7)$$

2. Assumptions

Throughout this section we write $\lesssim$ to indicate that particular constants are not important for the bounds.

Assumption 1 (Regularity of coefficients). *For all $s, t \in [0, T]$ and $x, y, z \in \mathbb{R}$:*

1. *The coefficients are globally Lipschitz in the space-time variables:*

$$|a(t, x) - a(s, y)| + |b(t, x) - b(s, y)| \lesssim (|t - s| + |x - y|),$$

$$|c(t, x, z) - c(s, y, z)| \lesssim (|t - s| + |x - y|) (1 \wedge |z|).$$

2. *The coefficients satisfy a linear growth condition in the state variable:*

$$|a(t, x)| + |b(t, x)| \lesssim (1 + |x|),$$

$$|c(t, x, z)| \lesssim (1 + |x|) (1 \wedge |z|).$$

Assumption 2 (Smoothness). *Fix an integer $m \geq 3$.*

1. *For all $t \in [0, T]$, $x \in \mathbb{R}$, $z \neq 0$ and integers $k, \ell \geq 0$ with $1 \leq k + \ell \leq m$:*

$$|\partial_x^k a(t, x)| + |\partial_x^k b(t, x)| + |\partial_x^k \partial_z^\ell c(t, x, z)| \lesssim 1.$$

2. *There exists a measurable function $q \in \bigcap_{p \geq 2} L^p(\nu)$ such that for every $k = 1, \dots, m$:*

$$|\partial_x^k c(t, x, z)| \lesssim q(z), \quad \text{for all } x \in \mathbb{R}, z \neq 0.$$

Assumption 3 (Kernel assumptions). *Recall the definition of the lower Pruitt functions (7). Then*

1. *$\nu(du)$ is absolutely continuous.*
2. *$N^\pm(r) \asymp \psi^{L,\pm}(1/r)$.*

Remark 1. Under Assumption 3.1 $N^\pm$ are strictly decreasing and $N^\pm(\tau^\pm(t)) = \frac{1}{t}$. Moreover, under Assumption 3.2 one can show that there exists $\alpha^\pm \in (0, 2)$ such that $\tau^\pm(t) \lesssim t^{1/\alpha^\pm}$. We also define

$$\alpha := \max(\alpha^-, \alpha^+). \quad (8)$$

3. Well-posedness and some implications

We define the operator $\mathcal{L}_t$ on $f \in C_b^2(\mathbb{R})$ by

$$\mathcal{L}_t f(x) = a(t, x) f'(x) + \frac{1}{2} b(t, x)^2 f''(x) + \int_{\mathbb{R}} [f(x + c(t, x, z)) - f(x) - c(t, x, z) f'(x)] \nu(dz). \quad (9)$$

Proposition 1. *Let $u(t, x) := \mathbb{E}[\Phi(X_T^{t,x})]$, where $\Phi \in C_b^3(\mathbb{R})$ and $X^{t,x}$ is the strong solution of the SDE (1). Then under Assumptions 1 and 2, the following statements hold:*

(i) $u \in C^{1,3}((0, T) \times \mathbb{R})$ and its derivatives satisfy the polynomial growth condition for some $p \geq 0$:

$$|\partial_x^k u(t, x)| \lesssim (1 + |x|^p). \quad (10)$$

(ii) The function u is the unique classical solution of the backward Kolmogorov PDE:

$$\partial_t u(t, x) + \mathcal{L}_t u(t, x) = 0, \quad u(T, x) = \Phi(x). \quad (11)$$

Proof. Under Assumption 1, the SDE (1) has a unique strong solution (Situ, 2005, Thm. 117), and standard moment estimates give the polynomial bound for the solution itself. By Assumption 2 and (Breton, & Privault, 2020, Prop. 4.1), the solution flow $x \mapsto X_s^{t,x}$ is three-times differentiable in x . Moreover, (Breton, & Privault, 2020, Thm. 5.1) gives the corresponding L^q bounds for the flow derivatives. Thus, for every $q \geq 2$,

$$\sup_{s \in [t, T]} \mathbb{E}[|X_s^{t,x}|^q] \lesssim 1 + |x|^q, \quad \sup_{s \in [t, T]} \mathbb{E}[|\partial_x^k X_s^{t,x}|^q] < \infty, \quad k = 1, 2, 3. \quad (12)$$

The moment bounds in (12) combined with the boundedness of $\Phi^{(k)}$ allow us to apply the dominated convergence theorem to justify differentiating under the expectation, as the moment bounds provide the necessary integrable majorant.

For $k = 1, 2, 3$, differentiation of the stochastic flow using the chain rule yields

$$\partial_x^k u(t, x) = \mathbb{E} \left[\sum_{m=1}^k \Phi^{(m)}(X_T^{t,x}) B_{k,m}(\partial_x X_T^{t,x}, \dots, \partial_x^{k-m+1} X_T^{t,x}) \right],$$

where $B_{k,m}$ are the exponential Bell polynomials given by the Faà di Bruno formula (Comtet, 1974, Chap. III, Thm. A). The boundedness of the derivatives of Φ , together with the moment bounds for the derivatives of the stochastic flow, gives the required polynomial growth estimates for $\partial_x^k u$ and confirms that $u(t, \cdot) \in C^3(\mathbb{R})$.

Having established the spatial regularity of u , we now prove its differentiability in time and show that it solves the backward Kolmogorov PDE using the semigroup approach. Let $(P_{s,t})_{0 \leq s \leq t \leq T}$ be the time-inhomogeneous two-parameter semigroup defined by $P_{s,t} f(x) = \mathbb{E}[f(X_t^{s,x})]$. Then we can then write $u(t, x) = P_{t,T} \Phi(x)$. By the Markov property, for a small $h > 0$ such that $t + h < T$, we have $P_{t,T} = P_{t,t+h} P_{t+h,T}$. This allows us to write:

$$u(t, x) = P_{t,t+h} P_{t+h,T} \Phi(x) = P_{t,t+h} u(t + h, x).$$

Rearranging this and dividing by h gives us an expression:

$$\frac{u(t + h, x) - u(t, x)}{h} = - \left(\frac{P_{t,t+h} - I}{h} \right) u(t + h, x). \quad (13)$$

Let us examine the r.h.s. of the above equation. Let $f \in C^2(\mathbb{R})$. By Itô's formula on $[t, t + h]$,

$$f(X_{t+h}^{t,x}) - f(X_t^{t,x}) = \int_t^{t+h} \mathcal{L}_s f(X_s^{t,x}) ds + M_{t,t+h},$$

where $M_{t,t+h}$ is a local martingale part with $\mathbb{E}[M_{t,t+h}] = 0$. Taking expectations and using $X_t = x$ gives us

$$P_{t,t+h} f(x) - f(x) = \mathbb{E} \left[\int_t^{t+h} \mathcal{L}_s f(X_s^{t,x}) ds \right].$$

If $f(s, \cdot)$ and its derivatives have polynomial growth, and the coefficients satisfy the growth and integrability conditions in Assumption 1, the operator $\mathcal{L}_t$ is bounded by a polynomial: $|\mathcal{L}_t f(x)| \lesssim (1 + |x|^q)$ for some $q \geq 0$. By applying the moment bounds from (12), we can use Fubini's theorem to interchange the expectation and the integral.

$$\frac{P_{t,t+h} f(x) - f(x)}{h} = \frac{1}{h} \int_t^{t+h} \mathbb{E}[\mathcal{L}_s f(X_s^{t,x})] ds. \quad (14)$$

Hence, by the dominated convergence and Lebesgue differentiation theorems, noting that $X_t^{t,x} = x$, we get

$$\frac{1}{h} \int_t^{t+h} \mathbb{E}[\mathcal{L}_s f(X_s^{t,x})] ds \xrightarrow{h \rightarrow 0} \mathbb{E}[\mathcal{L}_t f(X_t^{t,x})] = \mathcal{L}_t f(x),$$

Consequently,

$$\lim_{h \rightarrow 0} \frac{P_{t,t+h} f(x) - f(x)}{h} = \mathcal{L}_t f(x).$$

Using the limit results, together with $u(t + h, \cdot) \rightarrow u(t, \cdot)$, we obtain the backward Kolmogorov equation:

$$\partial_t u(t, x) = -\mathcal{L}_t u(t, x). \quad (15)$$

For uniqueness, let $v \in C^{1,3}$ be another classical solution and set $w := v - u$. By linearity, w solves $\partial_t w + \mathcal{L}_t w = 0$ with $w(T, x) = 0$. Applying Itô's formula to $w(s, X_s^{t,x})$ on $[t, T]$ gives a sum of an ordinary integral and a martingale term. Taking expectations eliminates the martingale term (as it has zero mean) and gives:

$$\mathbb{E}[w(T, X_T^{t,x})] - \mathbb{E}[w(t, X_t^{t,x})] = \mathbb{E}\left[\int_t^T (\partial_s w + \mathcal{L}_s w)(s, X_s^{t,x}) ds\right] = 0.$$

Using the terminal condition $w(T, \cdot) = 0$ and the initial condition $X_t^{t,x} = x$, this simplifies to $0 - w(t, x) = 0$. Thus, $w(t, x) = 0$ for all (t, x) , which implies $v \equiv u$. $\square$

Corollary 2. *If we additionally assume higher smoothness, namely $u \in C^{1,5}$, then under the hypotheses of Proposition 1, for every integer $0 \leq \ell \leq 3$ and for some $p \geq 0$, the mixed derivatives satisfy the growth condition*

$$|\partial_t \partial_x^\ell u(t, x)| \lesssim (1 + |x|^p).$$

Proof. From Proposition 1, we have the identity $\partial_t u = -\mathcal{L}_t u$. The regularity $u \in C^{1,5}$ allows us to differentiate with respect to x up to order ℓ :

$$\partial_t \partial_x^\ell u(t, x) = -\partial_x^\ell (\mathcal{L}_t u).$$

We bound the term on the right by analyzing its drift, diffusion and jump components separately.

The drift and diffusion components of the operator's derivative are

$$I_1^\ell := \partial_x^\ell [a(t, x) \partial_x u(t, x)], \quad I_2^\ell := \frac{1}{2} \partial_x^\ell [b^2(t, x) \partial_x^2 u(t, x)].$$

Applying the Leibniz rule gives the expansions

$$I_1^\ell = \sum_{k=0}^{\ell} C_\ell^k (\partial_x^k a) (\partial_x^{\ell-k+1} u), \quad I_2^\ell = \frac{1}{2} \sum_{k=0}^{\ell} C_\ell^k (\partial_x^k b^2) (\partial_x^{\ell-k+2} u),$$

where the term $\partial_x^k b^2$ expands to $\sum_{m=0}^k C_k^m (\partial_x^m b) (\partial_x^{k-m} b)$.

Proposition 1 yields polynomial growth $|\partial_x^m u| \lesssim (1 + |x|)^{p_0}$ for some $p_0 \geq 0$. For $j \geq 1$, Assumption 2 gives uniform bounds on the derivatives of the coefficients, $|\partial_x^j a| \leq C$ and $|\partial_x^j b| \leq C$; for $j = 0$, Assumption 1 gives $|a| + |b| \lesssim (1 + |x|)$. Hence, each term in the Leibniz sums is either bounded by $(1 + |x|)$ (from a) or $(1 + |x|)^2$ (from b^2). Consequently, for some $p_1 \geq 0$

$$|I_1^\ell(t, x)| + |I_2^\ell(t, x)| \lesssim (1 + |x|)^{p_1}.$$

Let $c := c(t, x, z)$ and $I_3^\ell(t, x) := \partial_x^\ell \int_{\mathbb{R}} J(t, x, z) \nu(dz)$, where by the Taylor theorem,

$$J(t, x, z) := u(t, x + c) - u(t, x) - c \partial_x u(t, x) = \frac{1}{2} c^2 \int_0^1 (1 - s) \partial_x^2 u(t, x + s c) ds.$$

We need to move ∂_x^ℓ inside the z -integral. To justify this step, we build an integrable majorant for $|\partial_x^\ell J|$. Using the Leibniz rule, we obtain

$$\partial_x^\ell J(t, x, z) = \frac{1}{2} \sum_{k=0}^{\ell} C_\ell^k (\partial_x^k c^2) \int_0^1 (1 - s) \partial_x^{\ell-k+2} u(t, x + s c) ds.$$

For $j = 0$, Assumption 1 provides the bound $|\partial_x^j c| \lesssim (1 + |x|)(1 \wedge |z|)$, while for $j = 1, 2, 3$, Assumption 2 gives $|\partial_x^j c| \lesssim q(z)$. Since each $\partial_x^j c^2$ is a product of at most two such entries, we have

$$|\partial_x^k c^2(t, x, z)| \lesssim (1 + |x|)^2 \left[(1 \wedge |z|)^2 + (1 \wedge |z|) q(z) + q(z)^2 \right].$$

Proposition 1 says that for $m \leq 5$ for some $p_2 \geq 0$

$$|\partial_y^m u(t, y)| \lesssim (1 + |y|)^{p_2}, \quad t \in [0, T], \quad y \in \mathbb{R}.$$

Because $|c| \lesssim (1 + |x|)$, replacing y by $x + s c$, $s \in [0, 1]$ shows that the s -integral is bounded by $(1 + |x|)^{p_3}$ for some $p_3 \geq 0$.

Combining the previous two steps, for some $p_4 \geq 0$ we obtain

$$|\partial_x^\ell J(t, x, z)| \lesssim (1 + |x|)^{p_4} G(z), \quad G(z) := (1 \wedge |z|)^2 + (1 \wedge |z|) q(z) + q(z)^2.$$

Next, we show that $G \in L^1(\nu)$. Indeed,

- (1) $\int_{\mathbb{R}} (1 \wedge |z|)^2 \nu(dz) < \infty$ based on the Lévy measure definition.
- (2) $\int_{\mathbb{R}} q(z)^2 \nu(dz) < \infty$ by Assumption 2.
- (3) Using the Cauchy-Schwarz inequality and letting $q \in L^2(\nu)$,

$$\int_{\mathbb{R}} (1 \wedge |z|) q(z) \nu(dz) \leq \left(\int_{\mathbb{R}} (1 \wedge |z|)^2 \nu(dz) \right)^{1/2} \left(\int_{\mathbb{R}} q(z)^2 \nu(dz) \right)^{1/2} < \infty.$$

Since $|\partial_x^\ell J| \leq (1 + |x|)^{p_4} G(z)$ with $G \in L^1(\nu)$, dominated convergence theorem allows us to move ∂_x^ℓ inside the integral in the definition of the $I_3^\ell(t, x)$, giving

$$|I_3^\ell(t, x)| \lesssim (1 + |x|)^{p_4}.$$

Adding the polynomial bounds for the drift, diffusion, and jump parts together yields the result. $\square$

Lemma 1. *Under Assumption 1, for any $p \geq 1$, the grid-point moments of the schemes are finite:*

$$\mathbb{E} \left[\max_{0 \leq i \leq n} |X_{t_i}^\tau|^p \right] < \infty.$$

Proof. Let the grid be $0 = t_0 < t_1 < \dots < t_n = T$. It is enough to control the moments at the grid points. Indeed,

$$\mathbb{E} \left[\max_{0 \leq i \leq n} |X_{t_i}^\tau|^p \right] \leq \sum_{i=0}^n \mathbb{E} [|X_{t_i}^\tau|^p].$$

The number of points n is finite. Thus, it is sufficient to show that $\mathbb{E} [|X_{t_i}^\tau|^p] < \infty$ for all i . We do this by induction. We have the base case $\mathbb{E} [|X_0|^p] < \infty$. For the inductive step, we derive a recursive bound for $A_i := \mathbb{E} [|X_{t_i}^\tau|^p]$.

Fix i and consider the scheme's update on $[t_i, t_{i+1}]$ with coefficients frozen at $(t_i, X_{t_i}^\tau)$:

$$X_{t_{i+1}}^\tau = X_{t_i}^\tau + a_i \Delta_i + b_i (B_{t_{i+1}} - B_{t_i}) + \int_{t_i}^{t_{i+1}} \int_{\mathbb{R} \setminus B(t_i, h, \varepsilon)} c_i(z) \tilde{N}(ds, dz) + \mathbb{I}_{\{\text{scheme 2}\}} \sigma_i \xi_i,$$

where the coefficients a_i, b_i, c_i are defined as frozen at time t_i .

Next, our goal is to bound $\mathbb{E} [|X_{t_{i+1}}^\tau|^p | \mathcal{F}_{t_i}]$. We can do this by applying the inequality

$$|x_1 + \dots + x_k|^p \leq C_p (|x_1|^p + \dots + |x_k|^p)$$

and considering each term from the right-hand side separately.

Using Assumption 1 to bound coefficients, the moment bounds for the Brownian terms, and the Novikov's theorem 6 for the compensated Poisson integral, we obtain the one-step conditional bound:

$$\mathbb{E} [|X_{t_{i+1}}^\tau|^p | \mathcal{F}_{t_i}] \leq |X_{t_i}^\tau|^p + C \Delta_i^{\beta_p} (1 + |X_{t_i}^\tau|^p), \quad \beta_p := \min(1, p/2).$$

Taking full expectations gives the following recursion:

$$A_{i+1} \leq (1 + C \Delta_i) A_i + C \Delta_i, \quad A_i := \mathbb{E} [|X_{t_i}^\tau|^p].$$

Since $A_0 < \infty$, by induction we have that $A_i < \infty$ for all i . Therefore,

$$\mathbb{E} \left[\max_{0 \leq i \leq n} |X_{t_i}^\tau|^p \right] < \infty.$$

$\square$

Throughout the following theorem and its proof we write $\mathbb{E}[\cdot]$ for $\mathbb{E}^x[\cdot]$ with non-random initial condition $X_0 = x$.

Theorem 3. *Let Assumptions 1–3 be satisfied and $\Phi \in C_b^3(\mathbb{R})$. Then the weak error between the true solution X_T of SDE (1) and the approximations X_T^τ from the Euler schemes (5) and (6) is of order n^{-1} :*

$$|\mathbb{E}[\Phi(X_T)] - \mathbb{E}[\Phi(X_T^\tau)]| \lesssim n^{-1}.$$

This rate is achieved when h is chosen as $h = n^{-\frac{\alpha}{\varepsilon(2-\alpha)}}$ for scheme (5), and $h = n^{-\frac{\alpha}{\varepsilon(3-\alpha)}}$ for scheme (6).

Proof. In the proof we rely on the approach outlined in (Bossy, & Maurer, 2025).

Our goal is to find an upper bound for

$$\left| \mathbb{E}[\Phi(X_t)] - \mathbb{E}[\Phi(X_t^\tau)] \right|, \quad \Phi \in C_b^3(\mathbb{R}). \quad (16)$$

Applying Itô's formula from Theorem 4 to $u(t, X_t)$, we obtain

$$u(t, X_t) = u(0, X_0) + \int_0^t \underbrace{(\partial_s u + \mathcal{L}_s u)}_{=0}(s, X_s) ds + M_t,$$

where M_t is the martingale part with zero expected value defined by

$$M_t := \int_0^t b(s, X_s) \partial_x u(s, X_s) dB_s + \int_0^t \int_{\mathbb{R}} \left[u(s, X_{s-} + c(s, X_{s-}, z)) - u(s, X_{s-}) \right] \tilde{N}(ds, dz).$$

Therefore, $\mathbb{E}[u(t, X_t)] = u(0, X_0)$.

Let $\eta(s) := \sup\{t_k \geq 0 \mid t_k \leq s\}$. Then, using Kolmogorov PDE (11), for DC schemes (5) and (6) we obtain

$$u(t, X_t^\tau) = u(0, X_0^\tau) + \int_0^t (\partial_s u + \mathcal{L}_s^\tau u)(s, X_s^\tau) ds + M_t^\tau = u(0, X_0^\tau) + \int_0^t (-\mathcal{L}_s u + \mathcal{L}_s^\tau u)(s, X_s^\tau) ds + M_t^\tau,$$

where M_t^τ is the martingale part with zero expected value defined below for both schemes respectively

$$M_t^{\tau,1} := \int_0^t \int_{\mathbb{R} \setminus B(\eta(s), h, \varepsilon)} \left[u(s, X_{s-}^{\tau,1} + c(\eta(s), X_{\eta(s)}^{\tau,1}, z)) - u(s, X_{s-}^{\tau,1}) \right] \tilde{N}(ds, dz) + \\ + \int_0^t b(\eta(s), X_{\eta(s)}^{\tau,1}) \partial_x u(s, X_s^{\tau,1}) dB_s,$$

$$M_t^{\tau,2} := \int_0^t \int_{\mathbb{R} \setminus B(\eta(s), h, \varepsilon)} \left[u(s, X_{s-}^{\tau,2} + c(\eta(s), X_{\eta(s)}^{\tau,2}, z)) - u(s, X_{s-}^{\tau,2}) \right] \tilde{N}(ds, dz) + \\ + \int_0^t b(\eta(s), X_{\eta(s)}^{\tau,2}) \partial_x u(s, X_s^{\tau,2}) dB_s + \int_0^t d(\eta(s), X_{\eta(s)}^{\tau,2}) \partial_x u(s, X_s^{\tau,2}) dW_s,$$

$$d^2(s, x) := \int_{B(s, h, \varepsilon)} c^2(s, x, z) \nu(dz).$$

Thus, on $[t_i, t_{i+1})$, the Gaussian interpolation uses $d(t_i, X_{t_i}^\tau)$, with the variance over the interval $\sigma_i^2(X_{t_i}^\tau) = \Delta_i d^2(t_i, X_{t_i}^\tau)$.

$$\mathcal{L}_t^{1,\tau} f(x) = a_i f'(x) + \frac{1}{2} b_i^2 f''(x) + \int_{\mathbb{R} \setminus B(t_i, h, \varepsilon)} [f(x + c_i(z)) - f(x) - c_i(z) f'(x)] \nu(dz), \quad (17)$$

$$\mathcal{L}_t^{2,\tau} f(x) = a_i f'(x) + \frac{1}{2} b_i^2 f''(x) + \int_{\mathbb{R} \setminus B(t_i, h, \varepsilon)} [f(x + c_i(z)) - f(x) - c_i(z) f'(x)] \nu(dz) + \quad (18)$$

$$+ \frac{1}{2} f''(x) \int_{B(t_i, h, \varepsilon)} c_i^2(z) \nu(dz), \quad (19)$$

where $a_i := a(t_i, X_{t_i}^\tau)$, $b_i := b(t_i, X_{t_i}^\tau)$, $c_i(z) := c(t_i, X_{t_i}^\tau, z)$.

Note that since $X_0^\tau = X_0 = x$, we have $u(0, X_0^\tau) = u(0, X_0)$. Then, by taking expectations, we get

$$\mathbb{E}[\Phi(X_t)] - \mathbb{E}[\Phi(X_t^\tau)] = \int_0^T \mathbb{E}[(\mathcal{L}_t - \mathcal{L}_t^\tau)u(t, X_t^\tau)] dt. \quad (20)$$

Since the coefficients of the frozen Euler interpolation are frozen on $[t_i, t_{i+1})$ at the left endpoint value, we analyze the operator difference at the fixed state $x = X_{t_i}^\tau$.

Setting $f(x) := u(t, x)$, this difference can be decomposed as

$$(\mathcal{L}_t - \mathcal{L}_t^\tau)f(x) = (A) + (B) + (C) + (D), \quad (21)$$

where for $t \in [t_i, t_{i+1})$

$$(A) := [a(t, x) - a(t_i, x)] f'(x),$$

$$(B) := \frac{1}{2} [b^2(t, x) - b^2(t_i, x)] f''(x),$$

$$(C) := \int_{\mathbb{R} \setminus B(t_i, h, \varepsilon)} [f(x + c(t, x, z)) - f(x + c(t_i, x, z)) - (c(t, x, z) - c(t_i, x, z)) f'(x)] \nu(dz),$$

$$(D_1) := \int_{B(t_i, h, \varepsilon)} [f(x + c(t, x, z)) - f(x) - c(t, x, z) f'(x)] \nu(dz). \quad (\text{Scheme 1})$$

For Scheme 2, write $(D_2) = (D_{2,1}) + (D_{2,2})$, where

$$(D_{2,1}) := \int_{B(t_i, h, \varepsilon)} \left[f(x + c(t, x, z)) - f(x) - c(t, x, z) f'(x) - \frac{1}{2} c^2(t, x, z) f''(x) \right] \nu(dz),$$

$$(D_{2,2}) := \frac{1}{2} f''(x) \int_{B(t_i, h, \varepsilon)} [c^2(t, x, z) - c^2(t_i, x, z)] \nu(dz).$$

Bound (A). Using Assumption 1 and Proposition 1, we obtain

$$|(A)| \lesssim |t - t_i| |f'(x)| \lesssim \Delta_i (1 + |x|^{p_a}), \quad \text{for some } p_a \geq 0. \quad (22)$$

Bound (B). By Assumption 1 we have

$$|b^2(t, x) - b^2(s, x)| = (b(t, x) - b(s, x))(b(t, x) + b(s, x)) \lesssim |t - s|(1 + |x|).$$

Thus, adding Proposition 1, we obtain

$$|(B)| \lesssim (t - t_i)(1 + |x|) |f''(x)| \lesssim \Delta_i (1 + |x|)(1 + |x|^p) \lesssim \Delta_i (1 + |x|^{p_b}), \quad \text{for some } p_b \geq 0. \quad (23)$$

Bound (C). Let

$$c := c(t, x, z), \quad c_i := c(t_i, x, z), \quad \delta := c - c_i.$$

Assumptions 1–2 give us the following bounds

$$|\delta| \lesssim |t - t_i| (1 \wedge |z|), \quad |c_i| \lesssim (1 + |x|) (1 \wedge |z|).$$

The first order Taylor expansion of $g(u) := f(x + u)$ at $u = c_i$ yields:

$$g(c) = g(c_i) + \delta g'(c_i) + \frac{1}{2} \delta^2 g''(c_i + \theta \delta), \quad \theta \in (0, 1).$$

Hence,

$$g(c) - g(c_i) - \delta g'(c_i) = \frac{1}{2} \delta^2 g''(c_i + \theta \delta).$$

Applying the mean-value theorem and Proposition 1, we get:

$$|g'(c_i) - g'(0)| \lesssim (1 + |x|^{p_1}) |c_i|.$$

The second term is bounded by Proposition 1. After that the argument $x + c_i + \theta \delta$ grows at most linearly with $|x|$ by Assumption 1. Consequently, we have the following polynomial bound:

$$|g''(c_i + \theta \delta)| = |\partial_x^2 u(t, x + c_i + \theta \delta)| \lesssim (1 + |x + c_i + \theta \delta|^{p_2}) \lesssim (1 + |x|^{p_2}).$$

Then, for $f(x) = u(t, x)$ we have

$$\begin{aligned} |f(x + c) - f(x + c_i) - \delta f'(x)| &\lesssim (1 + |x|^{p_1}) |\delta| |c_i| + \frac{1}{2} |\delta|^2 (1 + |x|^{p_2}) \lesssim (1 + |x|^{p_c}) (|\delta| |c_i| + |\delta|^2) \lesssim \\ &\lesssim (1 + |x|^{p_c}) (|t - t_i| (1 \wedge |z|^2) + |t - t_i|^2 (1 \wedge |z|^2)) \lesssim (1 + |x|^{p_c}) \Delta_i (1 \wedge |z|^2). \end{aligned}$$

Finally, by definition of the Lévy measure we get

$$|(C)| \lesssim \Delta_i (1 + |x|^{p_c}), \quad \text{for some } p_c \geq 0. \quad (24)$$

Bound (D). For each z there exists $\theta_1 = \theta_1(z) \in (0, 1)$ and $\theta_2 = \theta_2(z) \in (0, 1)$ such that the Taylor expansion for both Euler schemes can be written as following:

$$f(x + c) - f(x) - cf'(x) = \frac{1}{2} c^2 f''(x + \theta_1 c), \quad (25)$$

$$f(x + c) - f(x) - cf'(x) - \frac{1}{2} c^2 f''(x) = \frac{1}{6} c^3 f'''(x + \theta_2 c). \quad (26)$$

Using Assumption 1 and Proposition 1,

$$|f(x + c) - f(x) - cf'(x)| \lesssim (1 + |x|^{p+2}) |z|^2,$$

$$|f(x + c) - f(x) - cf'(x) - \frac{1}{2} c^2 f''(x)| \lesssim (1 + |x|^{p+3}) |z|^3.$$

Here, we used the fact that $(1 \wedge |z|) = |z|$, since we are in the small-jump region.

For Scheme 2, the term $(D_{2,1})$ is controlled by the cubic Taylor remainder:

$$\left| f(x + c) - f(x) - cf'(x) - \frac{1}{2} c^2 f''(x) \right| \lesssim (1 + |x|^{p+3}) |z|^3.$$

Therefore,

$$|D_{2,1}| \lesssim (1 + |x|^{p+3}) \int_{B(t_i, h, \varepsilon)} |z|^3 \nu(dz).$$

It remains to control $(D_{2,2})$. By Assumption 1,

$$|c^2(t, x, z) - c^2(t_i, x, z)| \leq |c(t, x, z) - c(t_i, x, z)| (|c(t, x, z)| + |c(t_i, x, z)|).$$

Using the Lipschitz and growth bounds for c , we get

$$|c^2(t, x, z) - c^2(t_i, x, z)| \lesssim |t - t_i|(1 + |x|)^2(1 \wedge |z|)^2.$$

Hence, for some $p \geq 0$,

$$|D_{2,2}| \lesssim |t - t_i|(1 + |x|^p) \int_{\mathbb{R}} (1 \wedge |z|)^2 \nu(dz) \lesssim \Delta_i(1 + |x|^p).$$

Completing the proof.

Taking expectations and using the grid-point moment bound from Lemma 1, and noting that $\Delta_i \leq \frac{T}{n} \lesssim n^{-1}$, we get:

$$\int_0^T \mathbb{E}[|(A)| + |(B)| + |(C)|] dt \lesssim n^{-1}.$$

In addition, for the (D) part we apply Theorem 5 and obtain

$$\int_0^T \mathbb{E}[|(D_1)|] dt \lesssim \frac{\tau^2(h^\varepsilon)}{h^\varepsilon},$$

and, for Scheme 2,

$$\int_0^T \mathbb{E}[|(D_{2,1})|] dt \lesssim \frac{\tau^3(h^\varepsilon)}{h^\varepsilon}, \quad \int_0^T \mathbb{E}[|(D_{2,2})|] dt \lesssim n^{-1}.$$

Consequently,

$$\int_0^T \mathbb{E}[|(D_2)|] dt \lesssim \frac{\tau^3(h^\varepsilon)}{h^\varepsilon} + n^{-1}.$$

Using the Remark 1 and taking $h = n^{-\frac{\alpha}{\varepsilon(2-\alpha)}}$ for scheme 1, and $h = n^{-\frac{\alpha}{\varepsilon(3-\alpha)}}$ for scheme 2, we get

$$|\mathbb{E}[\Phi(X_T)] - \mathbb{E}[\Phi(X_T^r)]| \lesssim n^{-1}.$$

□

Appendix

Theorem 4 (Itô's formula for a compensated Lévy-type SDE (Applebaum, 2009, Thm. 4.4.7)). *Let X_t be a solution to (1). Then, for any function $f \in C_b^2(\mathbb{R})$, the following holds a.s. for all $t \geq 0$:*

$$\begin{aligned} f(X_t) - f(X_0) &= \int_0^t a(s, X_s) f'(X_s) ds + \frac{1}{2} \int_0^t b(s, X_s)^2 f''(X_s) ds + \int_0^t b(s, X_s) f'(X_s) dB_s + \\ &+ \int_0^t \int_{\mathbb{R}} [f(X_{s-} + c(s, X_{s-}, z)) - f(X_{s-}) - c(s, X_{s-}, z) f'(X_{s-})] \nu(dz) ds + \\ &+ \int_0^t \int_{\mathbb{R}} [f(X_{s-} + c(s, X_{s-}, z)) - f(X_{s-})] \tilde{N}(ds, dz). \end{aligned}$$

We state the next theorem in the form adopted by our framework.

Theorem 5 (Small jumps moments (Knopova, & Platonov, 2025, Lem. A.3)). *Let X_t be a solution to (1). Then for any $p \geq 2$*

$$\mathbb{E} \left[\int_0^t \int_{B(s, h, \varepsilon)} |c(s, X_s, z)|^p \nu(dz) ds \right] \lesssim \frac{\tau^p(h^\varepsilon)}{h^\varepsilon}.$$

Theorem 6. (Kühn, & Schilling, 2023, Th. 4.20) *Let $(X_t)_{t \geq 0}$ be a one-dimensional stochastic process given by*

$$X_t = \int_0^t \int_{y \neq 0} F(s, y) \tilde{N}(dy, ds), \quad t \geq 0,$$

where F is a predictable stochastic process, and $\tilde{N} = N - \hat{N}$ is a compensated Poisson random measure with compensator $\hat{N}(dy, ds) = \nu(dy)ds$, where the measure ν satisfies $\int_{y \neq 0} \min\{1, |y|^2\} \nu(dy) < \infty$.

1. (Novikov, 1975) *If*

$$\mathbb{E} \left[\int_0^T \int_{y \neq 0} |F(s, y)| \nu(dy) ds \right] < \infty,$$

for some $T > 0$, then for any $a \in [1, 2]$ and $p \in [0, a] \subseteq [0, 2]$, we have

$$\mathbb{E} \left(\sup_{t \leq T} |X_t|^p \right) \leq C_{p,a} \mathbb{E} \left[\left(\int_0^T \int_{y \neq 0} |F(s, y)|^a \nu(dy) ds \right)^{p/a} \right]. \quad (27)$$

2. (Kunita, 2004, Th.2.11) For all $p \geq 2$, we have

$$\mathbb{E} \left(\sup_{t \leq T} |X_t|^p \right) \leq c_p \mathbb{E} \left[\left(\int_0^T \int_{y \neq 0} |F(s, y)|^2 \nu(dy) ds \right)^{p/2} \right] + c_p \mathbb{E} \left[\int_0^T \int_{y \neq 0} |F(s, y)|^p \nu(dy) ds \right]. \quad (28)$$

Discussion and conclusions

We studied weak approximation of time-inhomogeneous Lévy-driven SDEs, analyzing two Euler schemes: (i) small jumps are removed completely and (ii) small jump are approximated with a Gaussian term. Under the stated assumptions, we proved $\mathcal{O}(n^{-1})$ weak error when scaling hyperparameter h is chosen optimally.

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АНАЛІЗ СЛАБКОЇ ЗБІЖНОСТІ ДИНАМІЧНОЇ ЗРІЗКИ ДЛЯ ПРОЦЕСІВ ТИПУ ЛЕВІ

Отримано оцінки слабкої збіжності для схеми Ейлера – Маруями одновимірних СДР, керованих процесом Леві, у яких малі стрибки динамічно зрізаються залежно від часу. Великі стрибки моделюються точно, тоді як малі стрибки (i) або повністю ігноруються, або (ii) замінюються нормальною випадковою величиною з відповідною дисперсією. За стандартних умов Ліпшиця та гладкості коефіцієнтів і міри Леві ми доводимо, що слабка похибка обох схем має порядок $\mathcal{O}(n^{-1})$. Цей порядок досягається вибором гіперпараметра масштабування $h = n^{-\alpha/(\varepsilon(2-\alpha))}$ для варіанта (i) та $h = n^{-\alpha/(\varepsilon(3-\alpha))}$ для варіанта (ii).

Ключові слова: динамічна зрізка, процеси типу Леві, схема Ейлера – Маруями.

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