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CONTROLLABILITY OF PSEUDOPARABOLIC INTEGRO-DIFFERENTIAL SYSTEMS WITH MEMORY

The paper studies linear pseudoparabolic integro-differential equations with a Volterra-type integral term, which provide an effective mathematical framework for describing dynamic processes with memory and nonlocal interactions. Such mathematical models are characteristic of various areas of applied research, where the evolution of a system depends not only on current conditions but also on its previous state. Typical examples include the behavior of viscoelastic materials (such as polymers, biopolymers, and high-temperature metals), the processes of groundwater filtration and contaminant transport in porous media, as well as in problems related to the transport of heat and nutrients in tissues.

The aim of the study is to generalize known theoretical results to a broader class of pseudoparabolic equations by introducing additional terms into both the differential and integral parts of the operators. To achieve this goal, the methodology of a priori estimates in negative norms, developed by S. I. Lyashko and his colleagues, is employed. The proposed approach enables the investigation of well-posedness, generalized solvability, and issues of controllability and optimal control for systems with distributed parameters.

The main result of the study is the establishment of a priori estimates in negative norms that account for the new terms introduced into the pseudoparabolic integro-differential equation. A key distinction of this work from previous studies is that the obtained estimates are proven without imposing non-negativity assumptions on the lower-order terms of the differential operator. This substantially broadens the applicability of the theory, allowing the extension of theorems on the well-posedness of initial-boundary value problems, as well as results on controllability and the existence of optimal control, to a wider class of pseudoparabolic-type systems.

Key words: pseudoparabolic integro-differential equation, Volterra operator, memory effect, a priori estimates, well-posedness, generalized solvability, controllability, optimal control.

AMS 2020 classification: 35R09, 35K70, 93C20, 93C23, 93B05, 45K05, 35A01.

Introduction

Partial differential equations appear in the analysis of many physical phenomena. They serve as a fundamental tool for mathematical modeling, providing a framework for describing dynamical systems that evolve over time. While classical PDEs typically focus on processes governed by local interactions, there exists a broad class of equations that extend this perspective by incorporating terms accounting for nonlocal and memory effects.

Temporal nonlocality is widely observed in physics, biology, chemistry, ecology, materials engineering, and mechanism design. For example, in many physical and biological systems, complex models account for the interaction of chemical reactions, momentum transport, diffusion processes, and mechanical stresses (Vromans, van de Ven, & Muntean, 2019), and such phenomena are frequently modeled by PDEs with delay (see, e.g., (Guo, Feng, & Li, 2025; Zhang, & Li, 2025; Pokojovy, & Anikushyn, 2019, 2017; Liu et al., 2025; Wu, & Yin, 2025; Ezzinbi, & Staili, 2025; Salhi, 2025; Demchenko, Anikushyn, & Pokojovy, 2019)). Another important class of temporal nonlocal models arises from fractional time derivatives, which encode history-dependent effects through power-law memory kernels. Such equations naturally emerge in the modeling of anomalous diffusion, viscoelasticity, and heterogeneous transport processes. Several derivations have been proposed using stochastic time-change, continuous-time random walks, and homogenization techniques (see, e.g., (Metzler, & Klafter, 2000; Mainardi, 1996; Mainardi et al., 2006; Caputo, & Mainardi, 1971; Kubica, Ryszewska, & Yamamoto, 2020; Gal, & Warma, 2020; Jin, & Zhou, 2024)).

Beyond delay and fractional models, a broad and versatile framework for temporal nonlocality is provided by PDEs with memory. These systems incorporate hereditary effects, where prior states influence present dynamics through convolution terms involving suitable memory kernels. Such behavior is commonly observed in processes involving porous media, including groundwater flow, pollutant transport in soil, and the movement of fluids and gases through fractured materials (e.g., (Lyashko, 2002; Cueto-Felgueroso, & Juanes, 2009; Di, Shang, & Zheng, 2016; Jhangeer et al., 2024)). Memory effects also arise in viscoelastic materials, which are extensively used in construction and industrial manufacturing. For example, in carbon sequestration applications, the prior history of gas injection and deformation of rock formations can significantly influence storage efficiency. Moreover, biological systems often exhibit memory-driven transport of heat and nutrients in tissues, which plays a crucial role in medical procedures such as hyperthermia-based cancer treatments (Ragab et al., 2021).

These memory effects are especially prominent in the mechanical response of certain materials (Anikushyn, Lyashko, & Samosonok, 2024). In the modeling of viscoelastic substances, for instance, one typically arrives at equations of motion formulated as linear hyperbolic differential equations modified by a dissipative Volterra-type integral term (Conti, Gatti, & Pata, 2008; Cannarsa, & Sforza, 2011; Duvaut, & Lions, 1976; Hranishak, & Anikushyn, 2024; Anikushyn, 2024). Materials that demonstrate such viscoelastic characteristics include amorphous and semicrystalline polymers, biopolymers, metals exposed to high temperatures, and bituminous substances (Lakes, 1998).

Numerous applied mathematical models governed by linear integro-differential equations have been rigorously explored in classical monographs (Renardy, Hrusa, & Nohel, 1987; Fabrizio, & Morro, 1992), where the authors offer formal justifications for these formulations and analyze their mathematical behavior.

Over the past years, various formulations for linear and nonlinear *pseudoparabolic* integro-differential equations and their systems have attracted significant attention across multiple research communities. The scope of this progress is vast and extends beyond what can be concisely summarized here. We provide just a few examples.

Di *et al.* (Di, Shang, & Yu, 2020) considered a nonlinear integro-differential equation of pseudoparabolic type given by

$$u_t - \Delta u_t - \Delta u + \int_0^t g(t - \tau) \Delta u(\tau) d\tau = f(u)$$

where the Volterra-type integral term accounts for the memory behavior of the system. Under homogeneous Dirichlet boundary conditions, the authors demonstrated the existence of finite-time blow-up solutions for initial data with non-negative and positive energy levels.

In a related study (Liu, Kim, & Fang, 2025), Liu *et al.* investigated the well-posedness and blow-up behavior of a semilinear pseudoparabolic equation with a nonlinear convolution memory term. The integral operator considered in this setting commonly arises in models from filtration theory, population dynamics, and semiconductor physics. The authors established threshold conditions that distinguish between global existence and blow-up scenarios for initial states with low energy.

Maqbul & Raheem (Maqbul, & Raheem, 2020) analyzed a one-dimensional semilinear pseudoparabolic equation, where the temporal nonlocal operator was applied directly to the function u , which differs from the previously examined cases where Δu served as the argument of the integral term. The problem was studied under homogeneous integral conditions, and the Rothe time-discretization method was applied to establish the existence and uniqueness of strong solutions.

Further research has focused on developing numerical methods for solving initial-boundary value problems of this class. For instance, in (Wang, 2024), efficient two-grid finite element algorithms were proposed for two-dimensional nonlinear pseudo-parabolic integro-differential problems. Optimal error estimates were obtained for fully discrete schemes in the H_1 and L_p norms. The analysis relied on a combined temporal-spatial error splitting strategy.

Amar *et al.* (Amar et al., 2019) studied a degenerate pseudo-parabolic problem with memory

$$\begin{aligned} -\operatorname{div} \left(C \nabla u_t + A \nabla u + \int_0^t B(x, t - \tau) \nabla u d\tau \right) &= f, \\ u &= 0, x \in \partial\Omega, \\ x|_{t=0} &= u_0(x) \text{ on } \Omega. \end{aligned}$$

Here A, B, C are symmetric matrices, A is positive definite and C is only positive semi-definite. The authors obtained the existence and uniqueness result.

In most physical and biological systems, diffusion is positive as it represents the spontaneous spreading of particles, heat, or other quantities from regions of high concentration to regions of low concentration, driven by the natural tendency toward entropy maximization. From a mathematical perspective, a positive diffusion coefficient ensures that the corresponding equation is parabolic and dissipative, implying that perturbations decay smoothly over time and that solutions remain stable.

However, negative (or effective negative) diffusion may arise in certain contexts, such as active media, reaction-diffusion systems, phase separation processes, or linearizations around unstable states. In such cases, feedback mechanisms, energy input from external sources, or coupling between system components lead to local aggregation rather than dispersion. Stability can often be recovered through additional effects, including nonlinear responses, higher-order corrections, or memory terms. For instance, effective negative diffusivity appears in the continuum limits of discrete models describing social aggregation (Popescu, & Dietrich, 2004; Miller et al., 2024, 2025).

In the pseudo-parabolic context, typical physical models leading to such equations (e.g., second-order fluids, consolidation of clays, double-porosity diffusion, viscoelastic diffusion with small relaxation time, Sobolev-type corrections) usually yield a positive definite coefficient matrix $B = \{b_{ij}\}_{i,j=1}^n$ (defined later in (5)), corresponding to positive diffusivity. This is the standard and physically natural case (Cannon, & Lin, 1988).

A negative sign of the matrix B in the linear constant-coefficient setting corresponds to an energy input or anti-diffusion. Such behavior is usually physically unstable or non-physical for simple diffusion or filtration models. Negative diffusivity was considered, for example, in (Bertsch, Smarrazzo, & Tesei, 2013, 2016), where the authors analyzed the pseudo-parabolic regularization

$$u_t = \Delta \varphi(u) + \varepsilon \Delta [\psi(u)]_t,$$

associated with the forward-backward parabolic equation

$$u_t = \Delta \varphi(u).$$

The key feature of that model is a Perona-Malik-type nonlinearity $\varphi(u)$ whose derivative $\varphi'(u)$ changes sign, for some $\alpha > 0$:

$$\varphi'(u) > 0 \quad \text{for } u \in [0, \alpha), \quad \varphi'(u) < 0 \quad \text{for } u \in (\alpha, \infty).$$

The derivative $\varphi'(u)$ thus acts as an effective diffusion coefficient multiplying Δu in

$$\Delta \varphi(u) = \nabla \cdot (\varphi'(u) \nabla u) = \varphi''(u) |\nabla u|^2 + \varphi'(u) \Delta u.$$

Hence, regions where $\varphi'(u) < 0$ correspond to effective negative diffusivity, since the coefficient multiplying Δu becomes negative. Whenever $\varphi'(u) < 0$, the equation $u_t = \Delta\varphi(u)$ becomes backward parabolic and thus ill-posed in the classical sense. The pseudo-parabolic regularization term $\varepsilon \Delta[\psi(u)]_t$ compensates for this instability and restores well-posedness by introducing higher-order temporal smoothing.

Analyzing such complex systems, particularly those that may be ill-posed or lack classical stability, requires a mathematical framework that allows one to establish well-posedness and controllability. One suitable approach is the methodology of a priori inequalities in negative norms, developed by S.I. Lyashko and his colleagues, the principles of which are presented in the monograph (Lyashko, 2002). The application of this theory has proven effective for issues of well-posedness of initial-boundary value problems, optimal control, controllability of systems with distributed parameters, etc.

This methodology has been comprehensively developed and applied in a number of subsequent works, particularly (Lyashko, & Semenov, 2001; Nomirowskyi, & Anikushyn, 2011; Nomirowskyi, 2004; Lyashko, & Nomirowskii, 2003; Lyashko, Nomirowskii, & Sergienko, 2001; Lyashko, & Nomirowskyi, 2003; Nomirowskyi, 2006; Nomirowskii, 2007; Nomirowskyi, & Vostrikov, 2016; Tymchyshyn, & Nomirowskii, 2021). Later, it has also proven effective for application to Dirichlet problems for various classes of integro-differential equations with Volterra-type integral terms. In particular, hyperbolic integro-differential equations were studied in (Hranishak et al., 2025; Anikushyn, 2024; Anikushyn, Lyashko, & Samosonok, 2024; Hranishak, & Anikushyn, 2024), parabolic equations in (Hulianyt'skyi, & Anikushyn, 2014; Anikushyn, 2014, 2025), and pseudoparabolic type equations were presented in (Hulianyt'skyi, & Semenov, 2012; Anikushyn, & Andaral, 2025; Anikushyn, & Nazarchuk, 2024).

In the present paper, we consider a system described by a linear pseudo-parabolic integro-differential equation with a Volterra-type integral operator

$$a(x)u_t - \sum_{i,j=1}^n (a_{ij}(x)u_{tx_j})_{x_i} + b(x)u + \sum_{i=1}^n b_i(x)u_{x_i} - \sum_{i,j=1}^n (b_{ij}(x)u_{x_j})_{x_i} + \\ + \int_0^t \left(\sum_{i,j=1}^n (K_{ij}(x, t, \tau)u_{x_j})_{x_i} + \sum_{i=1}^n K_i(x, t, \tau)u_{x_i} + K(x, t, \tau)u \right) d\tau = f(x, t)$$

with Dirichlet-type conditions

$$u|_{x \in \partial\Omega} = 0, \quad u|_{t=0} = 0.$$

A specific case of this initial-boundary value problem ($K_i = K = b_i = 0$) was studied in (Anikushyn, & Nazarchuk, 2024), where a priori estimates in negative norms were established. Using the approach developed in (Lyashko, 2002), the authors proved the well-posedness of the problem and formulated a theorem on the existence of an optimal control for the corresponding system.

In (Hulianyt'skyi, & Semenov, 2012), a priori estimates in negative norms were also obtained for a structurally related pseudoparabolic integro-differential equation containing an additional Volterra-type term. These estimates served as the basis for proving existence and uniqueness theorems for generalized solutions of the associated boundary value problems. The study additionally addressed the problem of impulse-point controllability for systems governed by such equations.

In the present work, we extend the analysis of the pseudoparabolic integro-differential equation considered in (Anikushyn, & Nazarchuk, 2024) by introducing additional terms with coefficients b_i in the differential part of the operator, as well as kernel functions K_{ij} , K_i , and K in the integral operator. We derive a priori estimates in negative norms for this extended equation, with full consideration of the new terms. The assumptions on the coefficients are retained from (Anikushyn, & Nazarchuk, 2024): the self-adjointness of the differential part, the non-negative definiteness of the matrix $\{a_{ij}\}_{i,j=1}^n$, and the non-negativity of the function a .

The derived inequalities allow us to generalize the results of (Anikushyn, & Nazarchuk, 2024) and (Hulianyt'skyi, & Semenov, 2012) to a wider class of equations. This generalization covers well-posedness, generalized solvability, the optimal control problem, and controllability, while crucially extending the framework from (Anikushyn, & Andaral, 2025) by removing the assumption of non-negative definiteness for the matrix $\{b_{ij}\}_{i,j=1}^n$ and non-negativity for the function b .

1. Main notations and spaces

Let us consider the cylindrical domain $Q = \Omega \times (0, T)$, where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, and the linear integro-differential equation with a Volterra-type integral operator

$$\mathcal{L}u \equiv L_D u + L_I u = f(x, t). \quad (1)$$

Here, $u(x, t)$ is the unknown function describing the state of the system in the domain Q , and the operators L_D and L_I are given by

$$L_D u \equiv (\mathcal{A}u)_t + \mathcal{B}u, \quad (2)$$

$$L_I u \equiv \int_0^t \left(\sum_{i,j=1}^n (K_{ij}(x, t, \tau)u_{x_j})_{x_i} + \sum_{i=1}^n K_i(x, t, \tau)u_{x_i} + K(x, t, \tau)u \right) d\tau, \quad (3)$$

where

$$\mathcal{A}u \equiv - \sum_{i,j=1}^n (a_{ij}(x)u_{x_j})_{x_i} + a(x)u, \quad (4)$$

$$\mathcal{B}u \equiv - \sum_{i,j=1}^n (b_{ij}(x) u_{x_j})_{x_i} + \sum_{i=1}^n b_i(x) u_{x_i} + b(x)u. \quad (5)$$

We assume that the coefficients $\{a_{ij}\}_{i,j=1}^n, \{b_{ij}\}_{i,j=1}^n \subset C^1(\overline{\Omega})$, $a, b, b_i \in C(\overline{\Omega})$ satisfy, for all $x \in \Omega$,

$$a_{ij}(x) = a_{ji}(x), b_{ij}(x) = b_{ji}(x), a(x) \geq 0, \quad (6)$$

and there exists a constant $\alpha > 0$ such that, for all $x \in \Omega$ and $\xi_i \in \mathbb{R}$, $i = \overline{1, n}$,

$$\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq \alpha \sum_{i=1}^n \xi_i^2. \quad (7)$$

We also assume that the kernels $K_{ij}(x, t, \tau)$, $K_i(x, t, \tau)$ are continuously differentiable, and $K(x, t, \tau)$ is continuous. In particular, there exists a constant $M > 0$ such that $|K_{ij}(x, t, \tau)| < M$, $|K_i(x, t, \tau)| < M$, and $|K(x, t, \tau)| < M$ for all $x \in \Omega$ and $t, \tau \in [0, T]$. For simplicity, we additionally assume that all functions $|a_{ij}|, |b_{ij}|, |b_i|, a, |b|$ in $\overline{\Omega}$ are bounded by the same constant M .

The domain of the operator $\mathcal{L}$ is taken to be the space of functions infinitely differentiable in $\overline{Q}$ and satisfying homogeneous initial and Dirichlet boundary conditions

$$u|_{t=0} = 0, \quad (8)$$

$$u|_{x \in \partial\Omega} = 0. \quad (9)$$

We denote this set of functions by C_{BR}^∞ .

We define the space W_{BR} as the completion of C_{BR}^∞ with respect to the norm

$$\|u\|_W = \left(\int_Q u_t^2 + \sum_{i=1}^n u_{x_i}^2 dQ \right)^{1/2}. \quad (10)$$

The adjoint operator (in $L_2(Q)$) is defined as $\mathcal{L}^*v \equiv L_D^*v + L_I^*v$, where

$$L_D^*v \equiv -(\mathcal{A}v)_t + \mathcal{B}^*v,$$

$$L_I^*v \equiv \int_t^T \left(\sum_{i,j=1}^n (K_{ij}(x, t, \tau) v_{x_i})_{x_j} - \sum_{i=1}^n (K_i(x, \tau, t) v)_{x_i} + K(x, \tau, t) v \right) d\tau.$$

The domain of $\mathcal{L}^*$ is taken to be the space of functions infinitely differentiable in $\overline{Q}$ and satisfying

$$v|_{t=T} = 0, \quad (11)$$

$$v|_{x \in \partial\Omega} = 0. \quad (12)$$

We denote this set by C_{BR+}^∞ , and by W_{BR+} we denote its completion with respect to the norm from Equation (10).

Furthermore, we denote by H_{BR} and H_{BR+} the completions of C_{BR}^∞ and C_{BR+}^∞ , respectively, with respect to the norm

$$\|u\|_H = \left(\int_Q u^2 + \sum_{i=1}^n u_{x_i}^2 dQ \right)^{1/2}, \quad (13)$$

and by $W_{BR}^-, W_{BR+}^-, H_{BR}^-, H_{BR+}^-$ the corresponding negative spaces constructed with respect to the associated positive spaces relative to $L_2(Q)$.

Remark 1. It can be shown that the completions of C_{BR}^∞ and C_{BR+}^∞ with respect to the norm (13) coincide. Hence, $H_{BR} = H_{BR+}$ and $H_{BR}^- = H_{BR+}^-$.

2. Generalized solvability and controllability

Consider a pseudoparabolic-type system in which the state function u satisfies the equation

$$\mathcal{L}u = f + \mathcal{B}h, \quad f \in W_{BR+}^-, \quad (14)$$

where the operator $\mathcal{L}$ is defined according to (1)–(5), and $\mathcal{B}$ describes the action of the control h from the admissible set $\mathcal{U}$ of the control space $\mathcal{H}$ on the system. The function $u(x, t)$ is subject to the initial-boundary conditions (8), (9).

In the next section, we establish a priori estimates that play a key role in justifying the well-posedness and generalized solvability of the initial-boundary value problem, as well as in formulating the controllability problem for the obtained solutions. We first present these estimates in the form of the following theorem.

Theorem 1. *There exist constants $C_1 > 0$ and $C_2 > 0$ such that, for all $u \in C_{BR}^\infty$ and $v \in C_{BR+}^\infty$, the following inequalities hold*

$$\begin{aligned} C_1 \|u\|_{H_{BR}} &\leq \|\mathcal{L}u\|_{W_{BR+}^-} \leq C_2 \|u\|_{W_{BR}}, \\ C_1 \|v\|_{H_{BR+}} &\leq \|\mathcal{L}^*v\|_{W_{BR}^-} \leq C_2 \|v\|_{W_{BR+}}. \end{aligned}$$

Below, we present an approach to defining a generalized solution to problem (14), following the ideas of (Lyashko, 2002; Anikushyn, & Andaral, 2025). Other possible approaches to defining a generalized solution can also be found in the cited literature.

Definition 1. A function $u \in H_{BR}$ is called a generalized solution of problem (14) with the right-hand side $F = f + \mathcal{B}h \in W_{BR+}^-$ if there exists a sequence $u_i \in C_{BR}^\infty$, $i = 1, 2, \dots$, such that

$$\|u - u_i\|_{H_{BR}} \xrightarrow{i \rightarrow \infty} 0, \quad \|\mathcal{L}u_i - F\|_{W_{BR+}^-} \xrightarrow{i \rightarrow \infty} 0.$$

Applying Theorem 3, and Lemma 3 of (Lyashko, 2002, pp. 8,10), along with inequalities from Theorem 1, the following statement holds.

Theorem 2. For any function $F \in W_{BR+}^-$, there exists a unique solution $u \in H_{BR}$ of problem (14) in the sense of Definition 1. Moreover, there exists a constant C independent of F such that

$$\|u\|_{H_{BR}} \leq C \|F\|_{W_{BR+}^-}.$$

Following (Lyashko, & Semenov, 2001), we focus on asymptotic controllability.

Definition 2. System (14) is said to be asymptotically controllable in H_{BR} by the set of controls $\mathcal{U}$ if, for any $u^* \in H_{BR}$, there exists a sequence $h_i \in \mathcal{U}$, $i = 1, 2, \dots$, such that

$$\|u(h_i) - u^*\|_{H_{BR}} \xrightarrow{i \rightarrow \infty} 0,$$

where $u(h_i)$ is the generalized solution of equation (14) with the right-hand side $F = f + \mathcal{B}h_i \in W_{BR+}^-$.

Based on the a priori estimates from Theorem 1, the results of Theorem 2, and the approach of (Lyashko, 2002), we obtain the following result.

Theorem 3. If the set $\mathcal{B}(\mathcal{U})$ is dense in W_{BR+}^- , then system (14) is asymptotically controllable in the space H_{BR} by the set of admissible controls $\mathcal{U}$.

We now consider an application of Theorem 3 to the problem of controllability for a system with impulsive action. Suppose that the operator $\mathcal{B}$ is given by

$$\mathcal{B}h = \sum_{i=1}^s \delta(t - t_i) \otimes g_i(x), \quad t_i \in [0, T], \quad g_i(x) \in L_2(\Omega), \quad (15)$$

where the control is $h = \{(t_i, g_i(x))\}_{i=1}^s$, a finite sequence of elements from the set $\mathcal{H} = ([0, T] \times L_2(\Omega))^s$.

It is known (Lyashko, & Semenov, 2001), that the set of functions

$$\left\{ \sum_{i=1}^s \delta(t - t_i) \otimes g_i(x) \mid h \in \mathcal{H} \right\}$$

is dense in W_{BR+}^- . Therefore, we can formulate the following theorem.

Theorem 4. Let the state of the system be described by the solution of problem (14) with the operator (15) and the control space $\mathcal{H}$, and suppose that $\mathcal{B}(\mathcal{U})$ is dense in W_{BR+}^- . Then system (14) is asymptotically (impulsively) controllable in H_{BR} by the set $\mathcal{U}$.

It is worth noting that similar results on density, and hence controllability, can also be obtained for other types of generalized actions. In particular, let us consider the case of impulsive-pointwise control, as studied in (Lyashko, & Semenov, 2001; Lyashko, Semenov, & Sergienko, 2002).

Suppose that the domain Ω is cylindrical with respect to the variable x_1 , i.e., $\Omega = (0, 1) \times \Omega'$, $\Omega' \in \mathbb{R}^{n-1}$. And let the operator $\mathcal{B}$ is given by the following equation

$$\mathcal{B}h = \sum_{i=1}^s \delta(t - t_i) \otimes \delta(x_1 - x_{1,i}) \otimes g_i(x_2, \dots, x_n), \quad t_i \in [0, T], \quad x_{1,i} \in [0, 1], \quad g_i \in L_2(\Omega'), \quad (16)$$

where the control is $h = \{(t_i, x_{1,i}, g_i(x_2, \dots, x_n))\}_{i=1}^s$, a finite sequence of elements from the set $\mathcal{H} = ([0, T] \times [0, 1] \times L_2(\Omega'))^s$.

The set of functions

$$\left\{ \sum_{i=1}^s \delta(t - t_i) \otimes \delta(x_1 - x_{1,i}) \otimes g_i(x_2, \dots, x_n) \mid h \in \mathcal{H} \right\}$$

is dense in W_{BR+}^- (Hulianytskyi, & Semenov, 2012; Lyashko, 2002), which ensures the applicability of the asymptotic controllability theorems to this type of generalized action as well.

3. A priori inequalities in negative norms

We now proceed to prove the a priori inequalities from Theorem 1 for the operators $\mathcal{L}$ and $\mathcal{L}^*$. We shall assume that C may denote different constants where its exact value is immaterial.

Lemma 1. *There exists a constant $C > 0$ such that, for any function $u \in C_{BR}^\infty$, the following inequality holds:*

$$C\|u\|_{W_{BR}} \geq \|\mathcal{L}u\|_{W_{BR}^-}.$$

Proof. For any function $v \in C_{BR+}^\infty$, consider the scalar product $(L_D u, v)_{L_2(Q)}$. Applying the Cauchy-Schwarz and Poincaré inequalities to estimate the term containing b_i , we obtain

$$\begin{aligned} \left| \int_Q v \sum_{i=1}^n b_i u_{x_i} dQ \right| &\leq \int_Q \left| v \sum_{i=1}^n b_i u_{x_i} \right| dQ \leq M \left(\int_Q n v^2 dQ \int_Q \sum_{i=1}^n u_{x_i}^2 dQ \right)^{1/2} \leq \\ &\leq C_p^2 M \sqrt{n} \left(\int_Q v_i^2 dQ \right)^{1/2} \left(\int_Q \sum_{i=1}^n u_{x_i}^2 dQ \right)^{1/2} \leq C_p^2 M \sqrt{n} \|v\|_{W_{BR+}} \|u\|_{W_{BR}}. \end{aligned}$$

The remaining terms of the differential part of the problem operator are estimated as in (Lyashko, 2002).

Thus, for some constant C , we have

$$|(L_D u, v)_{L_2(Q)}| \leq C \|v\|_{W_{BR+}} \|u\|_{W_{BR}}. \quad (17)$$

Next, for any $v \in C_{BR+}^\infty$, we estimate $(L_I u, v)_{L_2(Q)}$:

$$\begin{aligned} |(L_I u, v)_{L_2(Q)}| &\leq \left| \int_Q \int_0^t \sum_{i,j=1}^n (K_{ij}(x, t, \tau) u_{x_i})_{x_j} d\tau v(x, t) dQ \right| + \\ &+ \left| \int_Q \int_0^t \sum_{i=1}^n K_i(x, t, \tau) u_{x_i} d\tau v(x, t) dQ \right| + \left| \int_Q \int_0^t K(x, t, \tau) u(x, \tau) d\tau v(x, t) dQ \right|. \end{aligned}$$

By repeatedly applying the Cauchy-Schwarz and Poincaré inequalities, and taking into account conditions (8), (9), we obtain the following estimate for the first term:

$$\begin{aligned} \left| \int_Q \int_0^t \sum_{i,j=1}^n (K_{ij}(x, t, \tau) u_{x_i})_{x_j} d\tau v(x, t) dQ \right| &= \left| \int_Q \int_0^t \sum_{i,j=1}^n K_{ij}(x, t, \tau) u_{x_i}(x, \tau) d\tau v_{x_j}(x, t) dQ \right| \leq \\ &\leq \sum_{i,j=1}^n \int_Q \int_0^t |K_{ij}(x, t, \tau) u_{x_i}(x, \tau) v_{x_j}(x, t)| d\tau dQ \leq \\ &\leq M \sum_{i,j=1}^n \int_Q |v_{x_j}(x, t)| \int_0^t |u_{x_i}(x, \tau)| d\tau dQ \leq MT \sum_{i,j=1}^n \left(\int_Q v_{x_j}^2 dQ \int_Q u_{x_i}^2 dQ \right)^{1/2} \leq \\ &\leq n M T C_p^2 \left(\int_Q \sum_{j=1}^n v_{x_j}^2 dQ \right)^{1/2} \left(\int_Q \sum_{i=1}^n u_{x_i}^2 dQ \right)^{1/2} \leq C \|v\|_{W_{BR+}} \|u\|_{W_{BR}}. \end{aligned}$$

For the second term, we have

$$\begin{aligned} \left| \int_Q \int_0^t \sum_{i=1}^n K_i(x, t, \tau) u_{x_i} d\tau v(x, t) dQ \right| &\leq \sum_{i=1}^n \int_Q \int_0^t |K_i(x, t, \tau) u_{x_i}(x, \tau) v(x, t)| d\tau dQ \leq \\ &\leq MT \left(\int_Q n v^2 dQ \right)^{1/2} \left(\sum_{i=1}^n \int_Q u_{x_i}^2 dQ \right)^{1/2} \leq \\ &\leq M T C_p^2 \sqrt{n} \left(\int_Q v_i^2 dQ \right)^{1/2} \left(\int_Q \sum_{i=1}^n u_{x_i}^2 dQ \right)^{1/2} \leq C \|v\|_{W_{BR+}} \|u\|_{W_{BR}}. \end{aligned}$$

For the third term, we obtain

$$\begin{aligned} \left| \int_Q \int_0^t K(x, t, \tau) u(x, \tau) d\tau v(x, t) dQ \right| &\leq \int_Q \int_0^t |K(x, t, \tau) u(x, \tau) v(x, t)| d\tau dQ \leq \\ &\leq MT \left(\int_Q v^2 dQ \right)^{1/2} \left(\int_Q u^2 dQ \right)^{1/2} \leq \end{aligned}$$

$$\leqslant MTC_p^2 \left(\int_Q v_i^2 dQ \right)^{1/2} \left(\int_Q u_i^2 dQ \right)^{1/2} \leqslant C \|v\|_{W_{BR+}} \|u\|_{W_{BR}}.$$

Hence, for some constant C , we have

$$|(L_I u, v)_{L_2(Q)}| \leqslant C \|v\|_{W_{BR+}} \|u\|_{W_{BR}}. \quad (18)$$

From (17) and (18), we arrive at the overall estimate

$$|(\mathcal{L}u, v)_{L_2(Q)}| = |(L_D u, v)_{L_2(Q)} + (L_I u, v)_{L_2(Q)}| \leqslant |(L_D u, v)_{L_2(Q)}| + |(L_I u, v)_{L_2(Q)}| \leqslant C \|v\|_{W_{BR+}} \|u\|_{W_{BR}},$$

for some constant C .

Since the set C_{BR+}^∞ is dense in the space W_{BR+} , dividing the last inequality by $\|v\|_{W_{BR+}}$ and taking the supremum over v yields the desired inequality. $\square$

The following lemma is proved in a similar way.

Lemma 2. *There exists a constant $C > 0$ such that, for any function $v \in C_{BR+}^\infty$, the following inequality holds:*

$$C \|v\|_{W_{BR+}} \geqslant \|\mathcal{L}^* v\|_{W_{BR}^-}.$$

Remark 2. The inequalities established in Lemmas 1 and 2 allow the operators $\mathcal{L}$ and $\mathcal{L}^*$ to be extended by continuity from their domains of definition to the spaces W_{BR} and W_{BR+} , respectively. These inequalities remain valid for all $u \in W_{BR}$ and $v \in W_{BR+}$, while preserving the previous notation for the extended operators.

To prove the following inequalities, we shall use the following auxiliary results, established in (Anikushyn, & Andaral, 2025) and (Hulianytskyi, & Anikushyn, 2014).

Lemma 3. *Let $f \in C([0, T])$ and suppose that, for any $t, \tau \in [0, T]$, the inequality $|K(t, \tau)| \leqslant M$ holds. Then, for any constant $c > 0$, the following estimate is valid:*

$$\left| \int_0^T \int_0^t K(t, \tau) e^{c\tau} f(\tau) d\tau f(t) dt \right| \leqslant M \sqrt{\frac{T}{c}} \int_0^T e^{ct} f^2(t) dt.$$

Lemma 4. *Let $f \in C^1([0, T])$, $f(T) = 0$, and suppose that, for any $t, \tau \in [0, T]$, the inequality $|K(t, \tau)| \leqslant M$ holds. Then, for any constant $c > 0$, the following estimate is valid:*

$$\left| \int_0^T \int_0^t K(t, \tau) e^{c\tau} f'(\tau) d\tau f(t) dt \right| \leqslant \frac{MT}{c} \int_0^T e^{ct} (f'(t))^2 dt.$$

Lemma 5. *There exists a constant $C > 0$ such that, for any function $u \in W_{BR}$, the following inequality holds:*

$$\|\mathcal{L}u\|_{W_{BR+}^-} \geqslant C \|u\|_{H_{BR}}.$$

Proof. Consider any function $u \in C_{BR}^\infty$. Let

$$v(x, t) = -e^{ct} \int_T^t e^{-2c\tau} u(x, \tau) d\tau. \quad (19)$$

As in (Anikushyn, & Nazarchuk, 2024), we have $v \in W_{BR+}$. Differentiating with respect to t , we see that

$$u(x, t) = u^{(1)}(x, t) + u^{(2)}(x, t),$$

where

$$u^{(1)}(x, t) = -e^{ct} v_t(x, t), \quad (20)$$

$$u^{(2)}(x, t) = ce^{ct} v(x, t). \quad (21)$$

The positive constant c will be specified later. We now show that, for some positive number C , the inequality holds:

$$(\mathcal{L}u, v)_{L_2(Q)} \geqslant C \|v\|_{W_{BR+}}^2.$$

We first consider the scalar product $(L_D u, v)_{L_2(Q)}$. By applying integration by parts, the boundary condition (8), and noting that the function v defined by (19) satisfies $v|_{t=T} = 0$, we obtain

$$\begin{aligned} \int_Q au_t v dQ &= \int_Q (auv)_t dQ - \int_Q av_t u dQ = \int_\Omega \int_0^T (auv)_t dt d\Omega - \int_Q auv_t dQ = \\ &= \int_\Omega ((auv)|_{t=T} - (auv)|_{t=0}) d\Omega - \int_Q auv_t dQ = - \int_Q auv_t dQ = \end{aligned}$$

$$= - \int_Q au^{(1)}v_t dQ - \int_Q au^{(2)}v_t dQ. \quad (22)$$

Taking into account (19) and condition (8), we have

$$\begin{aligned} \sum_{i,j=1}^n \int_Q ((a_{ij}u_{x_j})_{x_i}v)_t dQ &= \sum_{i,j=1}^n \int_\Omega \int_0^T ((a_{ij}u_{x_j})_{x_i}v)_t dt d\Omega = \\ &= \sum_{i,j=1}^n \int_\Omega ((a_{ij}u_{x_j}|_{t=T})_{x_i}v|_{t=T} - (a_{ij}u_{x_j}|_{t=0})_{x_i}v|_{t=0}) d\Omega = 0. \end{aligned}$$

Now, using the Gauss formula and the last equality, we obtain

$$\begin{aligned} - \int_Q \sum_{i,j=1}^n (a_{ij}u_{x_j})_{x_i}v dQ &= - \int_Q \sum_{i,j=1}^n ((a_{ij}u_{x_j})_{x_i}v)_t dQ + \int_Q \sum_{i,j=1}^n (a_{ij}u_{x_j})_{x_i}v_t dQ = \\ &= - \int_Q \sum_{i,j=1}^n a_{ij}u_{x_j}v_{x_i}t dQ = - \int_Q \sum_{i,j=1}^n a_{ij}u_{x_j}^{(1)}v_{x_i}t dQ - \int_Q \sum_{i,j=1}^n a_{ij}u_{x_j}^{(2)}v_{x_i}t dQ. \end{aligned} \quad (23)$$

Similarly, applying the Gauss formula in the standard way to the terms obtained from the operator $\mathcal{B}$, we arrive at

$$\begin{aligned} (L_D u, v)_{L_2(Q)} &= - \int_Q \sum_{i,j=1}^n a_{ij}u_{x_j}^{(1)}v_{x_i}t dQ - \int_Q \sum_{i,j=1}^n a_{ij}u_{x_j}^{(2)}v_{x_i}t dQ - \\ &- \int_Q au^{(1)}v_t dQ - \int_Q au^{(2)}v_t dQ + \int_Q \sum_{i,j=1}^n b_{ij}u_{x_j}^{(1)}v_{x_i} dQ + \int_Q \sum_{i,j=1}^n b_{ij}u_{x_j}^{(2)}v_{x_i} dQ + \\ &+ \int_Q \sum_{i=1}^n b_i u_{x_i}^{(1)}v dQ + \int_Q \sum_{i=1}^n b_i u_{x_i}^{(2)}v dQ + \int_Q bu^{(1)}v dQ + \int_Q bu^{(2)}v dQ. \end{aligned}$$

In (Anikushyn, & Nazarchuk, 2024) it was shown that, taking into account condition (6) and the Poincaré and Cauchy inequalities, the terms involving $u^{(1)}$ can be estimated as

$$- \int_Q au^{(1)}v_t dQ = \int_Q ae^{ct}v_t^2 dQ \geq 0, \quad (24)$$

$$- \int_Q \sum_{i,j=1}^n a_{ij}u_{x_j}^{(1)}v_{x_i}t dQ = \int_Q \sum_{i,j=1}^n a_{ij}e^{ct}v_{x_j}t v_{x_i}t dQ \geq \alpha \int_Q \sum_{i=1}^n e^{ct}v_{x_i}^2 t dQ, \quad (25)$$

$$\int_Q bu^{(1)}v dQ = \frac{1}{2} \int_\Omega bv^2|_{t=0} d\Omega + \frac{c}{2} \int_Q be^{ct}v^2 dQ \geq -\frac{MC_p}{2} \sum_{i=1}^n \int_\Omega v_{x_i}^2|_{t=0} d\Omega - \frac{cMC_p}{2} \sum_{i=1}^n \int_Q e^{ct}v_{x_i}^2 dQ, \quad (26)$$

$$\begin{aligned} \int_Q \sum_{i,j=1}^n b_{ij}u_{x_j}^{(1)}v_{x_i} dQ &= \frac{1}{2} \int_\Omega \sum_{i,j=1}^n b_{ij}v_{x_j}|_{t=0}v_{x_i}|_{t=0} d\Omega + \frac{c}{2} \int_Q \sum_{i,j=1}^n b_{ij}e^{ct}v_{x_j}v_{x_i} dQ \geq \\ &\geq -\frac{nM}{2} \sum_{i=1}^n \int_\Omega v_{x_i}^2|_{t=0} d\Omega - \frac{nMc}{2} \sum_{i=1}^n \int_Q e^{ct}v_{x_i}^2 dQ. \end{aligned} \quad (27)$$

For the term with b_i we apply integration by parts, obtaining

$$\begin{aligned} \int_Q \sum_{i=1}^n b_i u_{x_i}^{(1)}v dQ &= - \int_Q \sum_{i=1}^n b_i e^{ct}v_{x_i}t v dQ = - \int_Q \sum_{i=1}^n (b_i e^{ct}v_{x_i}v)_t dQ + \int_Q \sum_{i=1}^n (b_i e^{ct}v)_t v_{x_i} dQ = \\ &= \int_\Omega \sum_{i=1}^n b_i v_{x_i}|_{t=0}v|_{t=0} d\Omega + c \int_Q \sum_{i=1}^n b_i e^{ct}v_{x_i}v dQ + \int_Q \sum_{i=1}^n b_i e^{ct}v_{x_i}v_t dQ. \end{aligned} \quad (28)$$

Recalling that the constant M bounds all coefficients $|b|$, $\{|b_i|\}_{i=1}^n$, $\{|b_{ij}|\}_{i,j=1}^n$ in the domain Ω , and using the Poincaré and Cauchy inequalities, we estimate the terms in (28) as follows:

$$\begin{aligned} \int_\Omega \sum_{i=1}^n b_i v_{x_i}|_{t=0}v|_{t=0} d\Omega &\geq -M \int_\Omega \sum_{i=1}^n |v_{x_i}|_{t=0}v|_{t=0}| d\Omega \geq -\frac{M}{2} \sum_{i=1}^n \int_\Omega (v_{x_i}^2|_{t=0} + v^2|_{t=0}) d\Omega \geq \\ &\geq -\frac{M}{2} \sum_{i=1}^n \int_\Omega v_{x_i}^2|_{t=0} d\Omega - \frac{MC_p}{2} \sum_{i=1}^n \int_\Omega v_{x_i}^2|_{t=0} d\Omega, \end{aligned} \quad (29)$$

$$\begin{aligned} c \int_Q \sum_{i=1}^n b_i e^{ct} v_{x_i} v dQ &\geq -Mc \int_Q \sum_{i=1}^n e^{ct} |v_{x_i} v| dQ \geq -\frac{Mc}{2} \sum_{i=1}^n \int_Q e^{ct} (v_{x_i}^2 + v^2) dQ \geq \\ &\geq -\frac{Mc}{2} \sum_{i=1}^n \int_Q e^{ct} v_{x_i}^2 dQ - \frac{cMC_p}{2} \sum_{i=1}^n \int_Q e^{ct} v_{x_i}^2 dQ, \end{aligned} \quad (30)$$

$$\begin{aligned} \int_Q \sum_{i=1}^n b_i e^{ct} v_{x_i} v_t dQ &\geq -M \int_Q \sum_{i=1}^n e^{ct} |v_{x_i} v_t| dQ \geq -\frac{M}{2} \sum_{i=1}^n \int_Q e^{ct} \left(kv_{x_i}^2 + \frac{v_t^2}{k} \right) dQ \geq \\ &\geq -\frac{Mk}{2} \sum_{i=1}^n \int_Q e^{ct} v_{x_i}^2 dQ - \frac{MC_p}{2k} \sum_{i=1}^n \int_Q e^{ct} v_{x_i}^2 dQ. \end{aligned} \quad (31)$$

Here $k > 0$ is an arbitrary number to be chosen later.

Overall, taking into account (24)–(31), for the terms with $u^{(1)}$ we obtain

$$\begin{aligned} &-\int_Q \sum_{i,j=1}^n a_{ij} u_{x_j}^{(1)} v_{x_i} dQ - \int_Q a u^{(1)} v_t dQ - \int_Q \sum_{i,j=1}^n (b_{ij} u_{x_j}^{(1)})_{x_i} v dQ + \int_Q \sum_{i=1}^n b_i u_{x_i}^{(1)} v dQ + \int_Q b u^{(1)} v dQ \geq \\ &\geq \alpha \int_Q \sum_{i=1}^n e^{ct} v_{x_i}^2 dQ - \left(\frac{MC_p}{2} \sum_{i=1}^n \int_Q v_{x_i}^2|_{t=0} d\Omega + \frac{cMC_p}{2} \sum_{i=1}^n \int_Q e^{ct} v_{x_i}^2 dQ \right) - \\ &-\left(\frac{nM}{2} \sum_{i=1}^n \int_Q v_{x_i}^2|_{t=0} d\Omega + \frac{nMc}{2} \sum_{i=1}^n \int_Q e^{ct} v_{x_i}^2 dQ \right) - \left(\frac{M}{2} \sum_{i=1}^n \int_Q v_{x_i}^2|_{t=0} d\Omega + \frac{MC_p}{2} \sum_{i=1}^n \int_Q v_{x_i}^2|_{t=0} d\Omega \right) - \\ &-\left(\frac{Mc}{2} \sum_{i=1}^n \int_Q e^{ct} v_{x_i}^2 dQ + \frac{cMC_p}{2} \sum_{i=1}^n \int_Q e^{ct} v_{x_i}^2 dQ \right) - \left(\frac{Mk}{2} \sum_{i=1}^n \int_Q e^{ct} v_{x_i}^2 dQ + \frac{MC_p}{2k} \sum_{i=1}^n \int_Q e^{ct} v_{x_i}^2 dQ \right). \end{aligned} \quad (32)$$

We now turn to the terms with $u^{(2)}$. Integrating by parts with respect to t and applying the boundary condition (11), for the term in (22) we have

$$\begin{aligned} -\int_Q a v_t u^{(2)} dQ &= -c \int_Q a v_t e^{ct} v dQ = -\frac{c}{2} \int_Q (a e^{ct} v^2)_t dQ + \frac{c^2}{2} \int_Q a e^{ct} v^2 dQ = \\ &= -\frac{c}{2} \int_Q \int_0^T (a e^{ct} v^2)_t dt d\Omega + \frac{c^2}{2} \int_Q a e^{ct} v^2 dQ = \\ &= -\frac{c}{2} \int_Q (a e^{cT} v^2|_{t=T} - a v^2|_{t=0}) d\Omega + \frac{c^2}{2} \int_Q a e^{ct} v^2 dQ = \\ &= \frac{c}{2} \int_Q a v^2|_{t=0} d\Omega + \frac{c^2}{2} \int_Q a e^{ct} v^2 dQ. \end{aligned} \quad (33)$$

Since $a(x) \geq 0$, this expression is non-negative.

Similarly, for the corresponding term in (23), we apply integration by parts with respect to t . This yields

$$\begin{aligned} -c \int_Q \sum_{i,j=1}^n a_{ij} e^{ct} v_{x_j} v_{x_i} dQ &= \frac{c}{2} \int_Q \sum_{i,j=1}^n (a_{ij} e^{ct} v_{x_j} v_{x_i})_t dQ + \frac{c^2}{2} \int_Q \sum_{i,j=1}^n a_{ij} e^{ct} v_{x_j} v_{x_i} dQ = \\ &= \frac{c}{2} \int_Q \sum_{i,j=1}^n a_{ij} v_{x_j} |_{t=0} v_{x_i} |_{t=0} d\Omega + \frac{c^2}{2} \int_Q \sum_{i,j=1}^n a_{ij} e^{ct} v_{x_j} v_{x_i} dQ. \end{aligned}$$

Applying the uniform ellipticity condition (7), we obtain the estimate

$$-\int_Q \sum_{i,j=1}^n a_{ij} u_{x_j}^{(2)} v_{x_i} dQ \geq \frac{\alpha c}{2} \int_Q \sum_{i=1}^n v_{x_i}^2|_{t=0} d\Omega + \frac{\alpha c^2}{2} \int_Q e^{ct} \sum_{i=1}^n v_{x_i}^2 dQ. \quad (34)$$

The remaining terms involving $u^{(2)}$ are estimated directly using the Cauchy and Poincaré inequalities:

$$\int_Q b u^{(2)} v dQ = c \int_Q b e^{ct} v^2 dQ \geq -cMC_p \sum_{i=1}^n \int_Q e^{ct} v_{x_i}^2 dQ, \quad (35)$$

$$-\int_Q \sum_{i,j=1}^n (b_{ij} u_{x_j}^{(2)})_{x_i} v dQ = \int_Q \sum_{i,j=1}^n b_{ij} u_{x_j}^{(2)} v_{x_i} dQ = c \int_Q \sum_{i,j=1}^n b_{ij} e^{ct} v_{x_j} v_{x_i} dQ \geq -nMc \sum_{i=1}^n \int_Q e^{ct} v_{x_i}^2 dQ. \quad (36)$$

The term with b_i is estimated similarly to (30):

$$\int_Q \sum_{i=1}^n b_i u_{x_i}^{(2)} v \, dQ = c \int_Q \sum_{i=1}^n b_i e^{ct} v_{x_i} v \, dQ \geq -\frac{Mc}{2} \sum_{i=1}^n \int_Q e^{ct} v_{x_i}^2 \, dQ - \frac{cMC_p}{2} \sum_{i=1}^n \int_Q e^{ct} v_{x_i}^2 \, dQ. \quad (37)$$

In view of (33)–(37), for the terms containing $u^{(2)}$, we obtain

$$\begin{aligned} & - \int_Q \sum_{i,j=1}^n a_{ij} u_{x_j}^{(2)} v_{x_i} \, dQ - \int_Q a u^{(2)} v_t \, dQ - \int_Q \sum_{i,j=1}^n (b_{ij} u_{x_j}^{(2)})_{x_i} v \, dQ + \int_Q \sum_{i=1}^n b_i u_{x_i}^{(2)} v \, dQ + \int_Q b u^{(2)} v \, dQ \geq \\ & \geq \frac{\alpha c}{2} \int_Q \sum_{i=1}^n v_{x_i}^2|_{t=0} \, d\Omega + \frac{\alpha c^2}{2} \int_Q e^{ct} \sum_{i=1}^n v_{x_i}^2 \, dQ - cMC_p \sum_{i=1}^n \int_Q e^{ct} v_{x_i}^2 \, dQ - \\ & - nMc \sum_{i=1}^n \int_Q e^{ct} v_{x_i}^2 \, dQ - \left(\frac{Mc}{2} \sum_{i=1}^n \int_Q e^{ct} v_{x_i}^2 \, dQ + \frac{cMC_p}{2} \sum_{i=1}^n \int_Q e^{ct} v_{x_i}^2 \, dQ \right). \end{aligned} \quad (38)$$

Thus, adding Equations (32) and (38), we obtain

$$\begin{aligned} (L_D u, v)_{L_2(Q)} & \geq -\frac{cMC_p}{2} \sum_{i=1}^n \int_Q e^{ct} v_{x_i}^2 \, dQ - cMC_p \sum_{i=1}^n \int_Q e^{ct} v_{x_i}^2 \, dQ - \frac{nMc}{2} \sum_{i=1}^n \int_Q e^{ct} v_{x_i}^2 \, dQ - \\ & - nMc \sum_{i=1}^n \int_Q e^{ct} v_{x_i}^2 \, dQ - \frac{cMC_p}{2} \sum_{i=1}^n \int_Q e^{ct} v_{x_i}^2 \, dQ - \frac{Mc}{2} \sum_{i=1}^n \int_Q e^{ct} v_{x_i}^2 \, dQ - \\ & - \frac{cMC_p}{2} \sum_{i=1}^n \int_Q e^{ct} v_{x_i}^2 \, dQ - \frac{Mc}{2} \sum_{i=1}^n \int_Q e^{ct} v_{x_i}^2 \, dQ - \frac{Mc}{2} \sum_{i=1}^n \int_Q e^{ct} v_{x_i}^2 \, dQ - \\ & - \frac{MC_p}{2} \sum_{i=1}^n \int_Q v_{x_i}^2|_{t=0} \, d\Omega - \frac{nM}{2} \sum_{i=1}^n \int_Q v_{x_i}^2|_{t=0} \, d\Omega - \frac{M}{2} \sum_{i=1}^n \int_Q v_{x_i}^2|_{t=0} \, d\Omega - \\ & - \frac{MC_p}{2} \sum_{i=1}^n \int_Q v_{x_i}^2|_{t=0} \, d\Omega + \frac{\alpha c}{2} \int_Q \sum_{i=1}^n v_{x_i}^2|_{t=0} \, d\Omega + \frac{\alpha c^2}{2} \int_Q e^{ct} \sum_{i=1}^n v_{x_i}^2 \, dQ - \\ & - \frac{MC_p}{2k} \int_Q e^{ct} \sum_{i=1}^n v_{x_i}^2 \, dQ + \alpha \int_Q e^{ct} \sum_{i=1}^n v_{x_i}^2 \, dQ = \\ & = \frac{1}{2} (\alpha c^2 - Mc(5C_p + 3n + 2) - Mk) \int_Q e^{ct} \sum_{i=1}^n v_{x_i}^2 \, dQ + \\ & + \frac{1}{2} (\alpha c - M(2C_p + n + 1)) \int_Q \sum_{i=1}^n v_{x_i}^2|_{t=0} \, d\Omega + \left(\alpha - \frac{MC_p}{2k} \right) \int_Q e^{ct} \sum_{i=1}^n v_{x_i}^2 \, dQ. \end{aligned}$$

Choosing $k = \frac{MC_p}{\alpha}$ and a sufficiently large c such that $\alpha c \geq M(2C_p + n + 1)$, we have

$$(L_D u, v)_{L_2(Q)} \geq \frac{\alpha}{2} \int_Q e^{ct} \sum_{i=1}^n v_{x_i}^2 \, dQ + \frac{1}{2} \left(\alpha c^2 - Mc(5C_p + 3n + 2) - \frac{M^2 C_p}{\alpha} \right) \int_Q e^{ct} \sum_{i=1}^n v_{x_i}^2 \, dQ. \quad (39)$$

Next, consider the scalar product $(L_I u, v)_{L_2(Q)}$. Applying the Gauss formula to the terms containing $K_{ij}(x, t, \tau)$ and $K_i(x, t, \tau)$, and taking into account the initial-boundary conditions, we have

$$\begin{aligned} (L_I u, v)_{L_2(Q)} & = - \sum_{i,j=1}^n \int_Q \int_0^t K_{ij}(x, t, \tau) u_{x_i}^{(1)}(x, \tau) \, d\tau v_{x_i}(x, t) \, dQ - \sum_{i,j=1}^n \int_Q \int_0^t K_{ij}(x, t, \tau) u_{x_i}^{(2)}(x, \tau) \, d\tau v_{x_i}(x, t) \, dQ + \\ & + \sum_{i=1}^n \int_Q \int_0^t K_i(x, t, \tau) u_{x_i}^{(1)}(x, \tau) \, d\tau v(x, t) \, dQ + \sum_{i=1}^n \int_Q \int_0^t K_i(x, t, \tau) u_{x_i}^{(2)}(x, \tau) \, d\tau v(x, t) \, dQ + \\ & + \int_Q \int_0^t K(x, t, \tau) u^{(1)}(x, \tau) \, d\tau v(x, t) \, dQ + \int_Q \int_0^t K(x, t, \tau) u^{(2)}(x, \tau) \, d\tau v(x, t) \, dQ. \end{aligned}$$

Using Lemmas 3 and 4, together with the Cauchy-Schwarz and Poincaré inequalities for each $x \in \Omega$, we estimate the terms of the integral part of the operator.

For the terms with $u^{(1)}$, we obtain

$$\left| \sum_{i,j=1}^n \int_Q \int_0^t K_{ij}(x, t, \tau) e^{c\tau} v_{x_i \tau}(x, \tau) \, d\tau v_{x_j}(x, t) \, dQ \right| \leq \frac{nMT}{c} \int_Q e^{ct} \sum_{i=1}^n v_{x_i}^2 \, dQ, \quad (40)$$

$$\left| \sum_{i=1}^n \int_Q \int_0^t K_i(x, t, \tau) e^{c\tau} v_{x_i\tau}(x, \tau) d\tau v(x, t) dQ \right| \leq \frac{MTC_p}{c} \int_Q e^{ct} \sum_{i=1}^n v_{x_i t}^2 dQ, \quad (41)$$

$$\left| \sum_{i=1}^n \int_Q \int_0^t K(x, t, \tau) e^{c\tau} v_\tau(x, \tau) d\tau v(x, t) dQ \right| \leq \frac{MT}{c} \int_Q e^{ct} v_t^2 dQ \leq \frac{MTC_p}{c} \int_Q e^{ct} \sum_{i=1}^n v_{x_i t}^2 dQ. \quad (42)$$

For the terms with $u^{(2)}$, we have

$$\left| \sum_{i,j=1}^n \int_Q \int_0^t K_{ij}(x, t, \tau) e^{c\tau} cv_{x_i}(x, \tau) d\tau v_{x_j}(x, t) dQ \right| \leq nM\sqrt{T}c \int_Q e^{ct} \sum_{i=1}^n v_{x_i}^2 dQ, \quad (43)$$

$$\left| \sum_{i=1}^n \int_Q \int_0^t K_i(x, t, \tau) e^{c\tau} cv_{x_i}(x, \tau) d\tau v(x, t) dQ \right| \leq MC_p\sqrt{T}c \int_Q e^{ct} \sum_{i=1}^n v_{x_i}^2 dQ, \quad (44)$$

$$\left| \sum_{i=1}^n \int_Q \int_0^t K(x, t, \tau) e^{c\tau} cv(x, \tau) d\tau v(x, t) dQ \right| \leq cM\sqrt{\frac{T}{c}} \int_Q e^{ct} v^2 dQ \leq MC_p\sqrt{T}c \int_Q e^{ct} \sum_{i=1}^n v_{x_i}^2 dQ. \quad (45)$$

Combining (39) and (40)–(45), we obtain

$$\begin{aligned} (\mathcal{L}u, v)_{L_2(Q)} &= (L_D u, v)_{L_2(Q)} + (L_I u, v)_{L_2(Q)} \geq (L_D u, v)_{L_2(Q)} - |(L_I u, v)_{L_2(Q)}| \geq \\ &\geq \frac{\alpha}{2} \int_Q e^{ct} \sum_{i=1}^n v_{x_i t}^2 dQ + \frac{1}{2} \left(\alpha c^2 - Mc(5C_p + 3n + 2) - \frac{M^2 C_p}{\alpha} \right) \int_Q e^{ct} \sum_{i=1}^n v_{x_i}^2 dQ - \\ &- \frac{nMT}{c} \int_Q e^{ct} \sum_{i=1}^n v_{x_i t}^2 dQ - \frac{MTC_p}{c} \int_Q e^{ct} \sum_{i=1}^n v_{x_i t}^2 dQ - \frac{MTC_p}{c} \int_Q e^{ct} \sum_{i=1}^n v_{x_i t}^2 dQ - \\ &- nM\sqrt{T}c \int_Q e^{ct} \sum_{i=1}^n v_{x_i}^2 dQ - MC_p\sqrt{T}c \int_Q e^{ct} \sum_{i=1}^n v_{x_i}^2 dQ - MC_p\sqrt{T}c \int_Q e^{ct} \sum_{i=1}^n v_{x_i}^2 dQ = \\ &= \left(\frac{\alpha}{2} - \frac{MT}{c} (2C_p + n) \right) \int_Q e^{ct} \sum_{i=1}^n v_{x_i t}^2 dQ + \\ &+ \frac{1}{2} \left(\alpha c^2 - Mc(5C_p + 3n + 2) - 2M\sqrt{T}c(2C_p + n) - \frac{M^2 C_p}{\alpha} \right) \int_Q e^{ct} \sum_{i=1}^n v_{x_i}^2 dQ. \end{aligned}$$

Choosing c sufficiently large, we have

$$(\mathcal{L}u, v)_{L_2(Q)} \geq \frac{\alpha}{4} \int_Q e^{ct} \sum_{i=1}^n v_{x_i t}^2 dQ \geq C \|v\|_{W_{BR+}}^2. \quad (46)$$

Note that u and v satisfy the inequality

$$\|u\|_{H_{BR}}^2 \leq 2e^{2cT} (1 + c^2 C_p) \|v\|_{W_{BR+}}^2.$$

Therefore, in view of (46) and, in a standard way, applying the Schwarz inequality, we obtain

$$C_1 \|u\|_{H_{BR}} \|v\|_{W_{BR+}} \leq C \|v\|_{W_{BR+}}^2 \leq (\mathcal{L}u, v)_{L_2(Q)} \leq \|\mathcal{L}u\|_{W_{BR+}^-} \|v\|_{W_{BR+}},$$

for some constant C_1 . This completes the proof for all $u \in C_{BR}^\infty$. The statement of the lemma for all other functions $u \in W_{BR}$ is obtained by passing to the limit, as described in (Anikushyn, & Andaral, 2025). $\square$

Lemma 6. *There exists a constant $C > 0$ such that for any function $v \in W_{BR+}$ the following inequality holds:*

$$\|\mathcal{L}^* v\|_{W_{BR}^-} \geq C \|v\|_{H_{BR+}}.$$

Proof. The proof is analogous to that of Lemma 5, considering the auxiliary function

$$u(x, t) = e^{-ct} \int_0^t e^{2c\tau} v(x, \tau) d\tau$$

and using analogues of Lemmas 3 and 4. $\square$

Discussion and conclusions

We extend the analysis of pseudoparabolic integro-differential equations with Volterra-type terms to a wider class that includes additional terms with coefficients b_i in the differential operator $\mathcal{B}$ and kernels K_{ij} , K_i , and K in the integral part. A key contribution of this work is the establishment of a priori estimates in negative norms under the structural assumptions

of (Anikushyn, & Nazarchuk, 2024), namely the self-adjointness of the differential part, the non-negative definiteness of $\{a_{ij}\}_{i,j=1}^n$, and the non-negativity of a . Crucially, we achieve this generalization without requiring the non-negative definiteness of the differential operator $\mathcal{B}$. This significantly extends the framework presented in the recent work (Anikushyn, & Andaral, 2025), where such non-negativity was assumed, thus broadening the applicability of these results. These estimates enable us to extend the theorems on the well-posedness of the initial-boundary value problem and the controllability results from (Anikushyn, & Nazarchuk, 2024) and (Hulianytskyi, & Semenov, 2012).

Although this issue is not addressed in the present paper, the same a priori estimates make it possible to extend the result concerning the existence of an optimal control to our setting, in a manner analogous to that presented in (Anikushyn, & Nazarchuk, 2024).

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КЕРОВАНІСТЬ ПСЕВДОПАРАБОЛІЧНИМИ ІНТЕГРО-ДИФЕРЕНЦІАЛЬНИМИ СИСТЕМАМИ З ПАМ'ЯТТЮ

Досліджено лінійні псевдопараболічні інтегро-диференціальні рівняння з інтегральним членом типу Вольтерра, що забезпечують ефективний математичний апарат для опису динамічних процесів із пам'яттю та нелокальними взаємодіями. Такі математичні моделі є характерними для різних напрямів прикладних досліджень, де еволюція системи залежить не лише від поточного стану, але й від її попередніх станів. Типові приклади охоплюють поведінку в'язкопружних матеріалів (таких як полімери, біополімери та високоміцні метали за високих температур), процеси фільтрації підземних вод і перенесення домішок у пористих середовищах, а також задачі, пов'язані з перенесенням тепла й поживних речовин у тканинах.

Метою роботи є узагальнення відомих теоретичних результатів на ширший клас псевдопараболічних рівнянь шляхом введення додаткових членів як у диференціальну, так і в інтегральну частини операторів. Для досягнення цієї мети використано методику апріорних оцінок у негативних нормах, розроблену С. І. Ляшком та його колегами. Запропонований підхід дає змогу досліджувати коректність постановки задач, узагальнену розв'язність, а також питання керованості й оптимального керування для систем із розподіленими параметрами.

Основним результатом дослідження є встановлення апріорних оцінок у негативних нормах, що враховують нові члени, введені до псевдопараболічного інтегро-диференціального рівняння. Ключова відмінність цієї роботи від попередніх полягає в тому, що отримані оцінки доведено без накладання вимог невід'ємності на члени нижчого порядку диференціального оператора. Це суттєво розширює застосовність теорії, даючи можливість поширити теореми про коректність початково-крайових задач, а також результати щодо керованості й існування оптимального керування на ширший клас систем псевдопараболічного типу.

Ключові слова:

псевдопараболічне інтегро-диференціальне рівняння, оператор Вольтерра, ефект пам'яті, апріорні оцінки, коректність постановки, узагальнена розв'язність, керованість, оптимальне керування.

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