

UDC 519.2

DOI: <https://doi.org/10.17721/1812-5409.2026/1.35>

Nataliya SHCHESTYUK, PhD (Phys. & Math.), Assoc. Prof.
ORCID ID: 0000-0002-7652-8157
e-mail: n.shchestyuk@ukma.edu.ua
National University of Kyiv-Mohyla Academy, Kyiv, Ukraine

Dmytro SLUCHYNSKYI, PhD Student
ORCID ID: 0009-0003-4178-1608
e-mail: d.sluchynskyi@ukma.edu.ua
National University of Kyiv-Mohyla Academy, Kyiv, Ukraine

Fedir SERDUK, Master of Arts
ORCID ID: 0009-0001-5785-3480
e-mail: f.serdiuk@ukma.edu.ua
National University of Kyiv-Mohyla Academy, Kyiv, Ukraine

SUBDIFFUSIVE MODELS WITH LOCAL VOLATILITY FOR ILLIQUID MARKETS

The paper focuses on subdiffusive models with local volatility that extend classical approaches to capture illiquidity effects and trapping events through stochastic time changes.

The illiquidity effects mean lower trading volume, wider bid-ask spreads, greater price volatility, and long periods without any trading for risky assets. These effects occur during crisis periods that negatively affect financial activity, or in emerging markets where the number of participants and, thus, the number of transactions is relatively low. All models based on Brownian motion are perpetually moving. They are not adapted for modeling periods with motionless stock returns. Nevertheless, in statistical physics, there are subdiffusion processes that are well suited to describe the dynamics of underlying returns with trapping events. However, for modeling the dynamics of the financial market with illiquid effects, we need not only to describe stock prices, but also to propose a method for option evaluation. The main option pricing tool in this paper is a fractional extension of the Dupire equation, which gives the fair price for call and put options for subdiffusive underlying assets and takes into account local volatility. It is a forward partial differential equation in which the derivative with respect to time is replaced by a Dzerbayshan – Caputo ($D - C$) derivative. The form of the $D - C$ derivative and fractal Dupire equation depends upon the chosen Lévy subordinator. We apply the equation for subdiffusive in a Brownian setting in the case of the inverted tempered stable subordinator and demonstrate numerical results for real financial data.

Due to the proposed approaches and numerical algorithm, the investor gets tools that allow him to take into account both the market's illiquidity and local volatility.

Key words: option pricing, subdiffusion, local volatility, tempered stable subordinator.

AMS 2020 classification: 91G20, 91G60, 60G40.

Introduction

An illiquid market refers to the state of a stock, bond, or other asset that cannot easily and readily be sold or exchanged for cash without a substantial loss in value. It happens for crisis periods that negatively affect financial activity or for emerging markets in which the number of participants, and thus the number of transactions, is rather low. As a result, illiquid assets tend to have lower trading volume, wider bid-ask spreads, greater price volatility, and long periods without any trading. To account for the market's ability to accelerate or slow down during periods of varying trading intensity, fractal models of activity over time have been proposed, see for example (Heyde, 1999; Casteli, Leonenko, & Shchestyuk, 2017; Leonenko, Liu, & Shchestyuk, 2025), etc. In the context of this model, the "activity time" process can be interpreted as time over which market prices evolve, and it is often associated with trading volume or the flow of new price-sensitive information. However, these models and any other models based on Brownian motion are perpetually moving; they are not adapted for modeling periods with motionless stock returns.

Nevertheless, in statistical physics, there are subdiffusion processes that are well suited to describe the dynamics of underlying returns with trapping events (Metzler, & Klafter, 2000). Magdyarz was the first to propose a subdiffusion extension of the Bachelier and Black-Scholes models (see (Magdziarz, Orzeł, & Weron, 2011; Magdziarz, 2009)). He considered the standard diffusive process $S_t = S_{H_t}$, which is time-changed by some stochastic process H_t called the inverse subordinator ("hitting time"). In general, for a given subordinator G_t , the inverse subordinator H_t is defined as

$$H_t = \inf \{ \tau > 0 : G_\tau \geq t \}, \quad (1)$$

and interpreted as the first time at which G_t hits the barrier t . The subordinator G_t in (1) is generally Lévy process with Laplace transform in the following form:

$$E \left(e^{-\theta G_t} \right) = e^{-t \Psi(\theta)},$$

where $\Psi(\theta)$ is called Lévy exponent, which can be written in the following driftless form

$$\Psi(\theta) = \int_0^{+\infty} \left(1 - e^{-\theta x} \right) \tilde{\nu}(dx),$$

$\tilde{\nu}(dx)$ is an appropriate Lévy measure. By construction, the inverted process may be constant, and any process subordinated by H_t exhibits motionless periods. Magdziarz constructed the corresponding martingale measure, showed that

the model is incomplete, and derived the formulas for European put and call option prices using the martingale method (Magdziarz, Orzeł, & Weron, 2011; Magdziarz, 2009). A general result that implies that subdiffusive markets are generally incomplete is given in (Torricelli, 2020), Theorem 2. In (Scalas, & Toaldo, 2021), Theorem 5, a pseudo-differential equation that generalizes the Black and Scholes equation for the underlying time-changed geometric Brownian motion was derived. However, Magdziarz's approach has two drawbacks. Firstly, following the martingale method, the option price can be evaluated by computing an integral that includes the hitting time density. Thus, the density should admit a closed form. An alternative is to calculate the price using a Monte Carlo simulation. These solutions are explored in (Magdziarz, 2009) in a Brownian setting for α -stable subordinator, in (Shchestyuk, & Tyshchenko, 2025) for inverse Gaussian subordinator, and in (Shchestyuk, Drin, & Tyshchenko, 2024) for Gamma subordinator. Unfortunately, in the general case, the closed form of the hitting time density is unknown, and we need to look for other ways. Secondly, Magdziarz considered subdiffusive models in a Brownian setting with constant volatility. In fact, the realized volatility of an underlying asset actually varies with time and with the underlying itself, and it is worth taking into account.

The object of the research is a subdiffusive model with local volatility that extend classical approaches to capture illiquidity effects and trapping events through stochastic time changes. To price options, we use the Dupire fractal equation, which does not require a closed form for hitting time and takes into account local volatility. It is a forward partial differential equation in which the derivative with respect to time is replaced by a Dzerbayshan – Caputo (D – C) derivative. In (Hainaut, & Leonenko, 2021) was proved that option prices for subdiffusion can be found as solutions of the fractal Dupire equation. The form of the D – C derivative and fractal Dupire equation depends upon the chosen Lévy subordinator. In (Hainaut, & Leonenko, 2021), it was detailed for α -stable and Poisson subordinators in the jump-diffusion setting.

The aim of the paper is to propose the subdiffusion model in case of a Brownian setting for a *Tempered Stable Subordinator* $TS(\alpha, \lambda, c)$, develop a numerical method for the corresponding fractional Dupire equation, and apply the model to real market data for demonstration of its practical relevance.

The paper is organized as follows. In the second section, we consider the subdiffusive model for the dynamics of the illiquid market and formulate the proposition about the form of the fractal Dupire equation for option pricing in the case of the tempered stable subordinator. After that, we shortly discuss the issues of the existence of arbitrage opportunities and the completeness of the TSS subdiffusion market, the interpretation of the TSS parameters in light of the market price of risk. Beside this we mention that inverse Gaussian $IG(\delta, \gamma)$ subordinator can be considered as tempered stable Lévy subordinator $TS(1/2, \gamma^2, \delta)$ and demonstrate the form of the corresponding fractal Dupire equation in this case. The third section is devoted to the numerical approach and pricing of options for real financial data.

1. Subdiffusive option pricing via fractal Dupire equation

We assume that the subdiffusive market consists of at least three components: a risk-free bond, a risky asset, and an option. The idea of subdiffusion models is to replace the calendar time t in the risk-free bond motion and the classical geometrical Brownian motion (GBM) by some stochastic process H_t that represents the hitting time, defined as (1). Thus, the bond B_t and stock S_t are the solutions of the time-changed stochastic differential equations:

$$\begin{aligned} \frac{dB_{H_t}}{B_{H_t}} &= r dH_t, B_0 = 1, \\ \frac{dS_{H_t}}{S_{H_t}} &= \left(\mu + \frac{\sigma_{H_t}^2}{2} \right) dH_t + \sigma_{H_t} dB_{H_t}, \quad t > 0, \end{aligned}$$

where $\sigma_t = \sigma_{H_t}$ is local volatility.

To model option pricing, we use the fractional approach that was introduced in (Hainaut, & Leonenko, 2021) in a jump-diffusion setting, where the derivative for the time in the Dupire equation is replaced by a Dzerbayshan – Caputo derivative for jump-diffusion. The applications of anomalous diffusions for option pricing and the volatility term structure were considered in (Jacquier, & Torricelli, 2020; Torricelli, 2020). Option pricing in time-changed Lévy models with compound Poisson jumps was explored in (Ivanov, & Ano, 2018). The theory in these papers was detailed for inverse α -stable (Magdziarz, 2009; Magdziarz, Orzeł, & Weron, 2011; Hainaut, & Leonenko, 2021), Mittag – Leffler (Scalas, & Toaldo, 2021) and inverted Poisson processes (Hainaut, & Leonenko, 2021). We denote the option price as $C_H(T, k)$, where T is the time to maturity and the logarithmic strike price as $k = \log K$. We then replace the standard time derivative in the classical Dupire equation with a convolution-type derivative. Thus, the Fractal Dupire equation can be written in the form

$${}^{\Psi}D C_H(T, k) = -r \frac{\partial}{\partial k} C_H(T, k) + \frac{1}{2} \sigma^2(T, S_{H_t}) \frac{\partial^2}{\partial k^2} C_H(T, k), \quad (2)$$

where ${}^{\Psi}D u(t)$ is the convolution-type derivative for an absolutely continuous on $(0, \infty)$ function $u(t)$, called the Dzerbayshan – Caputo (D – C) derivative and defined as

$${}^{\Psi}D u(t) = \int_0^t \frac{\partial}{\partial t} u(t-s) \nu(s) ds. \quad (3)$$

In the (3) the tail of the Lévy measure, or kernel $\nu(s)$, is obtained as

$$\nu(s) ds = \int_s^\infty \tilde{\nu}(dx) ds,$$

where the Lévy measure $\tilde{\nu}(dx)$ for the subordinator G_t with Lévy exponent Ψ can be expressed through the inverse Laplace transform

$$\tilde{\nu}(dx) = \mathcal{L}^{-1} \left[\frac{d}{dx} \Psi(x) \right] (x).$$

We apply the equation (2) to diffusion in a Brownian setting with the inverse tempered stable subordinator $TS(\alpha, \lambda, c)$. The $TS(\alpha, \lambda, c)$ subordinator is a type of Lévy subordinator with Lévy exponent

$$\Psi(x) = c((\lambda + x)^\alpha - \lambda^\alpha), \quad (4)$$

where $0 < \alpha < 1$ is stability parameter, $\lambda > 0$ is tempering parameter and $c > 0$. The corresponding hitting time process H_t is called an inverse tempered stable subordinator (ITS). Its analytical properties are fully detailed in (Kumar, & Vellaisamy, 2015) and (Alrawashdeh et al., 2017).

Proposition 1. *The European call option price $C_H(T, k) = C_H(S, K, T, \sigma_H)$ for a time-changed version of the Brownian setting with local volatility, in case of an inverse tempered stable subordinator H_t is a solution of the fractional Dupire equation:*

$$c \frac{\alpha \lambda^\alpha}{\Gamma(1-\alpha)} \int_0^T \frac{\partial}{\partial T} C_H(T-s, k) \Gamma(\alpha, \lambda s) ds = -r \frac{\partial}{\partial k} C_H(T, k) + \frac{1}{2} \sigma^2(T, S_{H_t}) \frac{\partial^2}{\partial k^2} C_H(T, k), \quad (5)$$

with the initial condition $C_H(0, K) = (S_{H_0} - K)^+$. The European put option price is a solution of the same fractional Dupire equation but with initial conditions $P_H(0, K) = (K - P_{H_0})^+$.

Proof. Using (4) we compute

$$\frac{d\Psi(x)}{dx} = c \cdot \alpha (\lambda + x)^{\alpha-1},$$

and obtain Lévy measure:

$$\tilde{\nu}(dx) = c \frac{\alpha}{\Gamma(1-\alpha)} x^{-\alpha-1} e^{-\lambda x} I_{x>0} dx.$$

To compute the kernel $\nu(s)$, we evaluate the tail integral:

$$\nu(s) ds = \int_s^\infty \tilde{\nu}(dx) ds = c \frac{\alpha}{\Gamma(1-\alpha)} \int_s^\infty x^{-\alpha-1} e^{-\lambda x} dx ds.$$

Using the change of variables $x = \frac{t}{\lambda} \Rightarrow dx = \frac{dt}{\lambda}$ we have

$$\nu(s) ds = c \frac{\alpha \lambda^\alpha}{\Gamma(1-\alpha)} \int_{\lambda s}^\infty t^{-\alpha-1} e^{-t} dt ds = c \frac{\alpha \lambda^\alpha}{\Gamma(1-\alpha)} \Gamma(-\alpha, \lambda s) ds,$$

where $\Gamma(-\alpha, \lambda s)$ denotes the upper incomplete gamma function. Then D-C derivative (3) for $TS(\alpha, \lambda, c)$ subordinator can be written as

$${}^\Psi D C_H(T, k) = c \frac{\alpha \lambda^\alpha}{\Gamma(1-\alpha)} \int_0^T \frac{\partial}{\partial T} C_H(T-s, k) \Gamma(\alpha, \lambda s) ds. \quad (6)$$

Thus, if we substitute (6) in (2), the proposition follows. $\square$

We can notice that the derivative we obtain is a tempered D-C derivative (see (Sabzikar, Meerschaert, & Chen, 2015; Scalas, & Toaldo, 2021)). In general, these kinds of convolution operators have been introduced in (Kochubei, 2011) and then further explored in (Toaldo, 2015) and (Chen, 2017).

Remark 1. The issues of the existence of arbitrage opportunities and the completeness of the TSS subdiffusion market were investigated in (Torricelli, 2020; Toaldo, 2015), where was proved that subdiffusive TSS market is arbitrage-free and incomplete.

Remark 2. The presented option pricing subdiffusive model with $TS(\alpha, \lambda, c)$ subordinator has an interesting interpretation of its parameters in light of the market price of risk. Thanks to the theorem about the incompleteness of the subdiffusive market, the existence of a market price of duration risk naturally arises (see (Torricelli, 2020)). Once S_t is calibrated to liquid market prices, the market price of duration risk will be reflected in the martingale density parameters, and these same features will thus be reflected in the (risk-adjusted) frequency and duration of trade pauses incorporated in hitting time H_t . In (Torricelli, 2020) was shown the parameter λ expresses belief in the speed at which the granular price evolution of the asset will revert to a fully "liquid" state, which is well approximated by a standard Lévy-driven diffusion. So, λ captures the risk of latency to liquidity, while absolute trade duration risk is defined by α . Therefore, $TS(\alpha, \lambda, c)$ models offer a unified framework in which ultra short-term option pricing accounts for the microstructural duration effect, whereas options at typical maturities are priced according to the usual Lévy paradigm: the cut-off between the two states is encoded in λ .

Remark 3. Since the inverse Gaussian $IG(\delta, \gamma)$ subordinator can be considered as tempered stable Lévy subordinator $TS(1/2, \gamma^2, \delta)$, we can refer to the paper (Shchestsyuk, & Tyshchenko, 2025), where the fair option price for IG market was

explored as a solution of the PIDE:

$$2\delta \int_0^T \frac{\partial}{\partial t} C_H(T-s, k) \left(\gamma(\Phi(\gamma\sqrt{s}) - 1) + \frac{e^{-\frac{s\gamma^2}{2}}}{\sqrt{2\pi s}} \right) ds = -r \frac{\partial}{\partial k} C_H(T, k) + \frac{\sigma^2}{2} \frac{\partial^2}{\partial k^2} C_H(T, k),$$

with the initial condition $C_H(0, K) = (S_{H_0} - K)^+$, and $k = \ln K$. Here, parameter γ captures the risk of latency to liquidity, while absolute trade duration risk remains constant and is defined by $1/2$ (see (Shchestyuk, & Tyshchenko, 2025; Torricelli, 2020)).

Remark 4. If $\gamma = 0$ for $TS(1/2, \gamma^2, \delta)$, we get a regime where prices exhibit maximum staleness at all temporal scales. In fact,

$$\Psi_D C_H(T, k) = 2\delta \int_0^T \frac{\partial}{\partial T} C_H(T-s, k) \cdot \frac{1}{\sqrt{2\pi s}} ds.$$

Using the substitution $z = T - s \Rightarrow ds = -dz$, we get

$$\Psi_D C_H(T, k) = \frac{2\delta}{\sqrt{2\pi}} \Gamma\left(\frac{1}{2}\right) \frac{1}{\Gamma\left(1 - \frac{1}{2}\right)} \int_0^T \frac{\partial}{\partial z} C_H(z, k) (T-z)^{1/2} dz. \quad (7)$$

We remind the definition of the Caputo derivative

$$D_\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial}{\partial z} u(z) (t-z)^\alpha dz, \quad (8)$$

and use $\Gamma(1/2) = \sqrt{\pi}$, then we can rewrite (7) in the form

$$\Psi_D C_H(T, k) = \frac{2\delta}{\sqrt{2}} D_{1/2} C_H(T, k).$$

This leads to the fractional Dupire-type equation:

$$\frac{\delta}{\sqrt{2}} D_{1/2} C_H(T) = -\frac{r}{2} \frac{\partial}{\partial k} C_H(T, k) + \frac{\sigma^2(T, S_{H_t})}{4} \frac{\partial^2}{\partial k^2} C_H(T, k). \quad (9)$$

2. Numerical approach and illustration

In this section, a numerical approach involving the discretization of the integro-differential equation (9) is applied. The Caputo derivative on the left-hand side of (9) admits a finite difference representation (see, for example, (Deng, 2008)). In what follows, we use the discretization formula given in Eq. (37) of (Hainaut, & Leonenko, 2021):

$$\begin{aligned} \mathcal{D}_\alpha C_H(k, j) &\approx \frac{(\Delta t)^{-\alpha}}{\Gamma(1-\alpha)} (C_H(k, j) - C_H(k-1, j)) + \\ &+ \frac{(\Delta t)^{-\alpha}}{\Gamma(1-\alpha)} \sum_{m=0}^{k-2} (k-m)^{-\alpha} (C_H(m+1, j) - C_H(m, j)). \end{aligned}$$

The matrix form of Caputo derivative for the case $\alpha = \frac{1}{2}$ can be written as

$$\begin{aligned} \frac{1}{\sqrt{2}} \mathcal{D}_{1/2} C_H(k, \cdot)^T &\approx \frac{1}{\sqrt{2}} \frac{(\Delta t)^{-\frac{1}{2}}}{\sqrt{\pi}} (C_H(k, \cdot) - C_H(k-1, \cdot))^T + \\ &+ \frac{1}{\sqrt{2}} (D(k) C_H(0 : k-1, \cdot))^T \mathbf{1}_{k-1}, \end{aligned}$$

where $\mathbf{1}_{k-1}$ is a $(k-1)$ -vector of ones, $D(k)$ is a special matrix of dimension $(k-1) \times k$ with appropriate coefficients:

$$D(k) = \frac{(\Delta t)^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} \begin{pmatrix} 0 & \dots & \dots & 0 & -(2)^{-\frac{1}{2}} & (2)^{-\frac{1}{2}} \\ \vdots & \ddots & 0 & -(3)^{-\frac{1}{2}} & (3)^{-\frac{1}{2}} & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ -(k)^{-\frac{1}{2}} & (k)^{-\frac{1}{2}} & 0 & \dots & \dots & 0 \end{pmatrix}.$$

The first and second order derivatives with respect to k on the right side of (9) can be calculated by a central finite difference approximation:

$$\begin{aligned} \frac{\partial C_H(k, j)}{\partial k} &\approx \frac{C_H(k, j+1) - C_H(k, j-1)}{2\Delta_K}, \quad \frac{\partial C_H(k, 0)}{\partial k} \approx \frac{C_H(k, 1) - C_H(k, 0)}{\Delta_K}, \\ \frac{\partial C_H(k, n_K+1)}{\partial k} &\approx \frac{C_H(k, n_K+1) - C_H(k, n_K)}{\Delta_K}, \end{aligned}$$

$$\frac{\partial^2 C_H(k, j)}{\partial k^2} \approx \frac{C_H(k, j+1) - 2C_H(k, j) + C_H(k, j-1)}{\Delta_K^2},$$

$$\frac{\partial^2 C_H(k, 0)}{\partial k^2} \approx \frac{\partial^2 C_H(k, 1)}{\partial k^2}, \frac{\partial^2 C_H(k, n_K+1)}{\partial k^2} \approx \frac{\partial^2 C_H(k, n_K)}{\partial k^2},$$

where $n_K = \left\lceil \frac{K_{max} - K_0}{\Delta_K} \right\rceil$. These derivatives are represented in matrix form:

$$\frac{\partial C_H(k, j)}{\partial k} \approx \bar{K} R_1 C_H(k, \cdot)^T, \quad \frac{\partial^2 C_H(k, j)}{\partial k^2} \approx \bar{K}^2 R_2 C_H(k, \cdot)^T,$$

where

$$R_1 = \frac{1}{\Delta_K} \begin{pmatrix} -1 & 1 & 0 & \dots & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & \dots & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & -1 & 1 \end{pmatrix}, \quad R_2 = \frac{1}{\Delta_K^2} \begin{pmatrix} 1 & -2 & 1 & 0 & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & -2 & 1 \end{pmatrix},$$

$\bar{K} = \text{diag}(K_j)_{j \in 0, 1, \dots, n_K}$, the diagonal matrix of strikes and $C_H(k, \cdot)$ is the k -th line of the matrix of call prices, denoted by C_H (see (Hainaut, & Leonenko, 2021)).

Therefore, the final numerical iterative scheme for solving (9) in the case $\delta = 1$ can be written in the form:

$$C_H(k, \cdot)^T \approx \left(\bar{K} R_1 \frac{r}{2} - \bar{K}^2 R_2 \frac{\sigma_H^2}{4} + I_{n_K+1} \frac{1}{\sqrt{2}} \frac{(\Delta t)^{-\frac{1}{2}}}{\sqrt{\pi}} \right)^{-1} \times$$

$$\times \left(\frac{1}{\sqrt{2}} \frac{(\Delta t)^{-\frac{1}{2}}}{\sqrt{\pi}} (D(k) C_H(k-1, \cdot))^T \mathbf{1}_{k-1} \right),$$

where I_{n_K+1} is the identity matrix.

Numerical experiments were performed to compute European call option prices on AMD stock for the date 13/02/2025 with the following parameters: strike price $K = 111$, spot price $S = 111$, and interest rate $r = 5\%$. The annual constant volatility from historical data equals $\sigma = 47.37\%$. Local volatility σ_{local} was constructed iteratively using the recurrence relation:

$$\sigma_{\text{local}}(i) = \sigma_{\text{local}}(i-1) \cdot (\beta_0 + \beta_1 i^\gamma + \beta_2 i^{2\gamma}), \quad \text{for } i = 1, \dots, n_t,$$

with initial value $\sigma_{\text{local}}(0) = \sigma$ and parameters $\beta_0 = 0.99$, $\beta_1 = -1.10 \times 10^{-11}$, $\gamma = 1.63$, which was found by optimization procedure.

Three models were evaluated:

- Classical Black-Scholes model;
- Subdiffusive model with constant volatility;
- Subdiffusive model with local volatility.

Figure 1 presents the comparison of call option model prices and actual market prices (premiums) for varying times of maturity.

As we can see from the graphics in Fig. 1, the Black-Scholes option pricing model shows better results in the short-term period, while the subdiffusive model is more effective in the long-term perspective. To assess and compare the predictive performance of the models, for two time periods, the following accuracy metrics were calculated: **MAE** – Mean Absolute Error, **MSE** – Mean Squared Error, **RMSE** – Root Mean Squared Error, **ARPE** – Average Relative Pricing Error.

The results are presented in the Tab. 1.

Table 1

Accuracy metrics by time interval and model					
Interval	Model	MAE	MSE	RMSE	ARPE
2025-03-21 to 2025-09-19	Black-Scholes	0.753	0.652	0.807	0.061
	Subdiffusive (const)	3.888	15.533	3.941	0.369
	Subdiffusive (local)	3.087	10.218	3.197	0.302
2025-10-17 to 2026-01-16	Black-Scholes	1.066	1.140	1.068	0.054
	Subdiffusive (const)	2.128	4.656	2.158	0.109
	Subdiffusive (local)	0.888	0.974	0.987	0.046

Discussion and conclusions

The paper develops a subdiffusive model with local volatility time-changed by inverse TSS subordinator. The main pricing tool for European call and put options in this paper is a fractional extension of Dupire's equation in a Brownian setting with local volatility. We apply the equation for TSS -subordinator and demonstrate numerical results for real historical financial data.

The results of comparing the studied model with the classical one and real option prices show that the classical Black-Scholes model demonstrates better results in the short-term period, while the subdiffusive models are more effective in the long-term perspective. These results, based on the fractal Dupire equation technique, are similar to those obtained with the

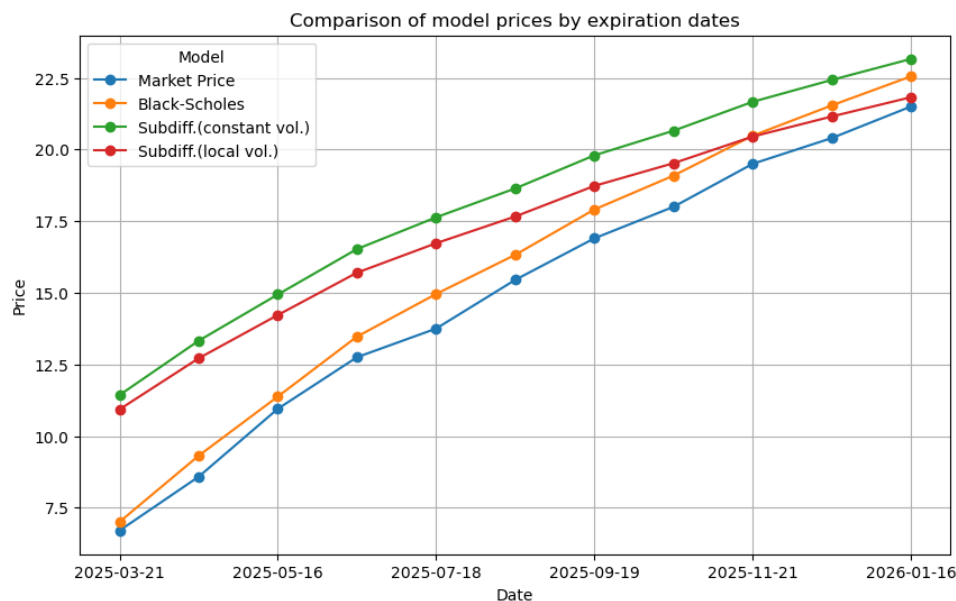


Fig. 1. Comparison of model call option prices with actual market premiums

$IG(\delta, \gamma)$ subordinator using the Monte Carlo method in the already published article (Shchestyuk, & Tyshchenko, 2025). Moreover, the accuracy of the subdiffusive model with local volatility prevails over the subdiffusive model with constant volatility.

It may be noted that in the articles of other authors dealing with option pricing based on subdiffusion (Magdziarz, Orzel, & Weron, 2011; Magdziarz, 2009; Hainaut, & Leonenko, 2021), the fair prices of options were not compared with real market prices (premiums).

Due to the proposed approaches and numerical algorithm, the investor gets tools that allow him to take into account both the market's illiquidity and local volatility.

Authors' contribution: Nataliya Shchestyuk – conceptualization, methodology, analysis of sources, preparation of literature review or theoretical foundations of research; Dmytro Sluchynskyi – methodology, software, empirical data collection and validation, empirical research, analysis of sources, preparation of literature review or theoretical foundations of research; Fedir Serduk – software, empirical research.

Acknowledgments. Nataliya Shchestyuk acknowledges N. Leonenko for some ideas developed in this article.

Sources of funding. Dmytro Sluchynskyi acknowledges financial support from the project "Platform for model based solutions to systemic complexity challenges: bridging leading research schools of Finland and Ukraine by Double Doctor Degree" funded by the TFK program of the Finnish National Agency for Education EDUF.

References

- Alrawashdeh, M., Kelly, J., Meerschaert, M., & Scheffler, H. (2017). Applications of inverse tempered stable subordinators. *Computers and Mathematics with Applications*, 73, 892–905. <https://doi.org/10.1016/j.camwa.2016.07.026>
- Casteli, F., Leonenko, N., & Shchestyuk, N. (2017). Student-like models for risky asset with dependence. *Stochastic Analysis and Applications*, 35, 452–464. <https://doi.org/10.1080/07362994.2016.1266945>
- Chen, Z.-Q. (2017). Time fractional equations and probabilistic representation. *Chaos, Solitons and Fractals*, 102, 168–174.
- Deng, W. (2008). Finite element method for the space and time fractional Fokker-Planck equation. *SIAM Journal on Numerical Analysis*, 47, 204–226. <https://doi.org/10.1137/080714130>
- Hainaut, D., & Leonenko, N. (2021). Option pricing in illiquid markets: A fractional jump–diffusion approach. *Journal of Computational and Applied Mathematics*, 381, . <https://doi.org/10.1016/j.cam.2020.112995>
- Heyde, C. (1999). A risky asset model with strong dependence through fractal activity time. *Journal of Applied Probability*, 36, 1234–1239.
- Ivanov, R., & An, K. (2018). Option pricing in time-changed Lévy models with compound poisson jumps. *Modern Stochastics: Theory and Applications*, 6(1), 81–107.
- Jacquier, A., & Torricelli, L. (2020). Anomalous diffusions in option prices: connecting trade duration and the volatility term structure. *SIAM Journal on Financial Mathematics*, 11(4), 1137–1167.
- Kochubei, A. N. (2011). General fractional calculus, evolution equations, and renewal processes. *Integral Equations and Operator Theory*, 71(4), 583–600.
- Kumar, A., & Vellaisamy, P. (2015). Inverse tempered stable subordinators. *Statistics and Probability Letters*, 103, 134–141. <https://doi.org/10.1016/j.spl.2015.04.010>
- Leonenko, N., Liu, A., & Shchestyuk, N. (2025). Student models for a risky asset with dependence: Option pricing and greeks. *Austrian Journal of Statistics*, 54(1), 138–165. <https://doi.org/10.17713/ajs.v54i1.1952>
- Magdziarz, M. (2009). Black–Scholes formula in subdiffusive regime. *Journal of Statistical Physics*, 136, 553–564. <https://doi.org/10.1007/s10955-009-9791-4>

- Magdziarz, M., Orzel, S., & Weron, A. (2011). Option pricing in subdiffusive Bachelier model. *Journal of Statistical Physics*, 145, 187–202. <https://doi.org/10.1007/s10955-011-0310-z>
- Metzler, R., & Klafter, J. (2000). The random walk's guide to anomalous diffusion: a fractional dynamics approach. *Physics Reports*, 339(1), 1–77. [https://doi.org/10.1016/S0370-1573\(00\)00070-3](https://doi.org/10.1016/S0370-1573(00)00070-3)
- Sabzikar, F., Meerschaert, M., & Chen, J. (2015). Tempered fractional calculus. *Journal of Computational Physics*, 293, 14–28.
- Scalas, E., & Toaldo, B. (2021). Limit theorems for prices of options written on semi-Markov processes. *Theory of Probability and Mathematical Statistics*, 105, 3–33.
- Shchestsyuk, N., Drin, S., & Tyshchenko, S. (2024). Risk evaluating for subdiffusive option price model with gamma subordinator. In *Mathematical and statistical methods for actuarial sciences and finance: Maf 2024, Le Havre, France* (pp. 286–291). Springer, Cham. https://doi.org/10.1007/978-3-031-64273-9_47
- Shchestsyuk, N., & Tyshchenko, S. (2025). Subdiffusive option price model with inverse Gaussian subordinator. *Modern Stochastics: Theory and Applications*, 12(2), 136–152. <https://doi.org/10.15559/24-VMSTA265>
- Toaldo, B. (2015). Convolution-type derivatives, hitting-times of subordinators and time-changed C_0 -semigroups. *Potential Analysis*, 42, 115–140. <https://doi.org/10.1007/s11118-014-9426-5>
- Torricelli, L. (2020). Trade duration risk in subdiffusive financial models. *Physica A: Statistical Mechanics and its Applications*, 541, 123694. <https://doi.org/10.1016/j.physa.2019.123694>

Отримано редакцією журналу / Received: 04.08.25

Прорецензовано / Revised: 18.02.26

Схвалено до друку / Accepted: 16.04.26

Опубліковано / Published: 05.06.26

Наталія ЩЕСТИЮК, канд. фіз.-мат. наук, доц.

ORCID ID: 0000-0002-7652-8157

e-mail: n.shchestsyuk@ukma.edu.ua

Національний університет «Києво-Могилянська академія», Київ, Україна

Дмитро СЛУЧИНСЬКИЙ, асп.

ORCID ID: 0009-0003-4178-1608

e-mail: d.sluchynskiy@ukma.edu.ua

Національний університет «Києво-Могилянська академія», Київ, Україна

Федір СЕРДЮК, магістр

ORCID ID: 0009-0001-5785-3480

e-mail: f.serdiuk@ukma.edu.ua

Національний університет «Києво-Могилянська академія», Київ, Україна

СУБДИФУЗІЙНІ МОДЕЛІ З ЛОКАЛЬНОЮ ВОЛАТИЛЬНІСТЮ ДЛЯ НЕЛІКВІДНИХ РИНКІВ

Розглянуто субдифузійні моделі з локальною волатильністю, що розширюють класичні підходи з метою врахування ефектів неліквідності ринку та затримок (пасток) цін шляхом стохастичних змін часу.

Ефекти неліквідності означають менший обсяг торгів, ширші спреди між цінами попиту та продажу, більшу волатильність цін і тривалі періоди без будь-якої торгівлі ризикованими активами. Ці ефекти виникають у кризові періоди, які негативно впливають на фінансову активність, або на ринках, що розвиваються, на яких кількість учасників, а отже, і кількість транзакцій, досить низька. Оскільки усі моделі, засновані на броунівському процесі, описують постійний неперервний хаотичний рух, то вони не пристосовані для моделювання цін акцій зі сталими періодами, коли ціни можуть не змінюватись протягом певного часу. Одним із варіантів виходу з такої ситуації є те, що у статистичній фізиці існують процеси субдифузії, які добре підходять для опису динаміки базових активів з ефектами потрапляння частинки в пастку, де вона може перебувати певний час без руху. Однак для моделювання динаміки фінансового ринку з ефектами неліквідності нам потрібно не лише описати ціни акцій, а й запропонувати метод оцінки опціонів. Основним інструментом ціноутворення опціонів у цій статті є дробове (фрактальне) розширення рівняння Дюпіра, що дає справедливу ціну для опціонів типу кол та пут у випадку субдифузійних базових активів і враховує локальну волатильність. Це диференціальне рівняння з частинними похідними, в якому похідна відносно часу замінюється фрактальною похідною Дзербайшана – Капуто ($D - C$). Вид похідної $D - C$ та фрактального рівняння Дюпіра залежить від обраного субординатора Леві. У цій статті ми застосовуємо зазначене рівняння для геометричної броунівської субдифузії у випадку оберненого TS (tempered stable) субординатора та демонструємо числові результати для реальних фінансових даних.

Завдяки запропонованим підходам і числовому алгоритму інвестор отримує інструменти, що дають йому можливість враховувати як неліквідність ринку, так і локальну волатильність.

Ключові слова: оцінка опціонів, субдифузія, локальна волатильність, темперований стабільний субординатор.

Автори заявляють про відсутність конфлікту інтересів. Спонсори не брали участі в розробленні дослідження; у зборі, аналізі чи інтерпретації даних; у написанні рукопису; в рішенні про публікацію результатів.

The authors declare no conflicts of interest. The funders had no role in the design of the study; in the collection, analyses or interpretation of data; in the writing of the manuscript; in the decision to publish the results.