

АЛГЕБРА, ГЕОМЕТРІЯ ТА ТЕОРІЯ ЙМОВІРНОСТЕЙ

UDC 515.47

DOI: <https://doi.org/10.17721/1812-5409.2026/1.2>

ANURAG, Master's Student
ORCID ID: 0009-0003-8391-7622
e-mail: anurag_kumar23@mru.ac.in
Manav Rachna University, Faridabad, India

Aparna VYAS, PhD, Prof.
ORCID ID: 0000-0002-5301-8234
e-mail: aparna@mru.edu.in
Manav Rachna University, Faridabad, India

Gibak KIM, PhD, Prof.
ORCID ID: 0000-0001-5114-4117
e-mail: imkgb27@ssu.ac.kr
Soongsil University, Seoul, Republic of Korea

EXTENDED UNIQUE FIXED POINT THEOREM IN COMPLETE QUASI-METRIC SPACE

The primary goal of this research is to present novel fixed point theorems within the framework of complete quasi-metric spaces. Fixed point theory is a critical area connecting pure and applied mathematics. While classical results like the Banach contraction principle are well-established in metric spaces, this paper extends these concepts to the more generalized quasi-metric space, which lacks the symmetric condition of a standard metric space. We prove a unique fixed point theorem for continuous mappings in a complete quasi-metric space under a newly defined, comprehensive contractive condition. This result significantly generalizes and extends several existing fixed-point theorems in the literature. An example is provided to illustrate the applicability of our findings. Our work contributes to the development of fixed point theory in asymmetric spaces, which have applications in diverse fields such as shape-memory alloys and Hamilton - Jacobi equations.

Key words: fixed point, contraction mapping, complete metric space, complete quasi metric space.

AMS 2020 classification: 47H10, 54H25.

Introduction

Fixed point theory is a fascinating branch of mathematics with deep connections across pure and applied disciplines. Its origins trace back to the early 20th century, notably with Banach's pioneering work in 1922 on the Banach contraction principle (Banach, 1922). This principle has wide-ranging applications in areas like differential and integral equations, metric spaces, and numerical analysis. Inspired by this, many researchers have proven various fixed point theorems in different mathematical spaces.

Following Banach, Kannan in 1968 (Kannan, 1968) and Chatterjea in 1972 (Chatterjea, 1972) introduced similar results on metric spaces, which were later generalized for complete metric spaces by Jojar et al. (Jojar et al., 2022). A pivotal development in the generalization of metric spaces was the introduction of quasi-metric spaces, also known as asymmetric metric spaces (Wilson, 1931). These spaces are defined without the symmetric condition of a standard metric, making them suitable for modeling phenomena where directional distance matters.

The relevance of quasi-metric spaces extends far beyond theoretical mathematics, finding numerous recent and significant applications in both pure and applied domains. These applications include, but are not limited to, the modeling of complex behaviors in shape-memory alloys (Mielke, & Roubíček, 2003), addressing questions of existence and uniqueness in Hamilton-Jacobi equations (Mennucci, 2013), facilitating automated taxonomy construction (Woon, & Madnick, 2009), providing robust models for material failure (Rieger, & Zimmer, 2009), and analyzing rate-independent models for plasticity (Mainik, & Mielke, 2005). The utility of fixed point theorems in these contexts often lies in proving the existence and uniqueness of solutions to equations that arise from such models.

This research article aims to further advance fixed point theory by establishing new fixed point theorems for continuous mappings in complete quasi-metric spaces. Our results introduce a novel contractive condition that extends and generalizes several well-known existing results in the literature. We provide a suitable example to demonstrate the applicability of our findings. This work contributes to the theoretical understanding of fixed points in asymmetric spaces and their potential use in various scientific and engineering problems.

1. Preliminaries

In this section, we introduce some basic definitions and an example that will be used further.

Definition 1. (Wilson, 1931): Let X be a nonempty set and let $d : X \times X \rightarrow [0, \infty)$ be a function satisfying the following conditions:

1. $d(x, y) = d(y, x) = 0 \implies x = y$;
2. $d(x, y) \leq d(x, z) + d(z, y) \quad \forall x, y, z \in X$.

Then, d is called quasi-metric (or q-metric) on X and the ordered pair (X, d) is called a quasi-metric space.

Example. (Ayoob, Chuan, & Mlaiki, 2023) An example of a quasi-metric space is a circle, where the distance d between two points A & B on the circle is defined as the length of the shortest arc measured in a clockwise direction.

Definition 2. (Jleli, & Samet, 2018) A sequence $\{x_n\}$ in a quasi-metric space (X, d) is said to be q -convergent to z if $\lim_{n \rightarrow \infty} d(x_n, z) = 0 = \lim_{n \rightarrow \infty} d(z, x_n)$. Here, z is the q -limit point of a sequence $\{x_n\}$.

Definition 3. (Jleli, & Samet, 2018) A sequence $\{x_n\}$ in a quasi-metric space (X, d) is said to be Cauchy sequence if for a given $\epsilon > 0$, there exists a $n_0 \in \mathbb{N}$ such that for all $m, n \geq n_0$,

$$d(x_n, x_m) < \epsilon \text{ or } d(x_m, x_n) < \epsilon.$$

Definition 4. (Wilson, 1931) A quasi-metric space (X, d) is considered complete if every Cauchy sequence in X converges to a point in X .

Definition 5. (Banach, 1922) Let (X, d) be a metric space and let $f : X \rightarrow X$ be a function. Then a point $x \in X$ is a fixed point of f if $f(x) = x$.

2. Main Results

In this section, we establish some fixed point theorems for continuous mappings in a complete quasi-metric space. Our results extend and generalize some well-known existing results in the literature in (Banach, 1922; Chatterjea, 1972; Jojar et al., 2022; Kannan, 1968; Mittal et al., 2023).

Theorem 1. Let (X, d) be a complete quasi-metric space and $T : X \rightarrow X$ be a continuous self-mapping satisfying the condition

$$\begin{aligned} d(Tx, Ty) \leq & a_1 d(x, y) + a_2 [d(x, Tx) + d(y, Ty)] + a_3 [d(x, Ty) + d(y, Tx)] + \\ & + a_4 \left[\frac{d(x, y)d(x, Ty)}{d(x, y) + d(y, Ty)} \right] + a_5 \left[\frac{d(y, Ty)(1 + d(x, Tx))}{1 + d(x, y)} \right] + a_6 \left[\frac{d(x, Ty)d(y, Ty)}{d(x, y) + d(y, Ty)} \right] \end{aligned} \quad (1)$$

for all $x, y \in X$, $a_1, a_2, a_3, a_4, a_5, a_6 \geq 0$ and $a_1 + 2a_2 + 2a_3 + a_4 + a_5 + a_6 < 1$. Thus, T has a unique fixed point in X .

Proof. Let $x_0 \in X$ be an arbitrary element in X and define an iterative sequence as

$$x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, \dots, x_n = Tx_{n-1} = T^n x_0$$

for all $n \in \mathbb{N}$. Then

$$\begin{aligned} d(x_1, x_2) &= d(Tx_0, Tx_1) \leq \\ &\leq a_1 d(x_0, x_1) + a_2 [d(x_0, Tx_0) + d(x_1, Tx_1)] + a_3 [d(x_0, Tx_1) + d(x_1, Tx_0)] + \\ &+ a_4 \left[\frac{d(x_0, x_1)d(x_0, Tx_1)}{d(x_0, x_1) + d(x_1, Tx_1)} \right] + a_5 \left[\frac{d(x_1, Tx_1)(1 + d(x_0, Tx_0))}{1 + d(x_0, x_1)} \right] + a_6 \left[\frac{d(x_0, Tx_1)d(x_1, Tx_1)}{d(x_0, x_1) + d(x_1, Tx_1)} \right] = \\ &= a_1 d(x_0, x_1) + a_2 [d(x_0, x_1) + d(x_1, x_2)] + a_3 [d(x_0, x_2) + d(x_1, x_1)] + \\ &+ a_4 \left[\frac{d(x_0, x_1)d(x_0, x_2)}{d(x_0, x_1) + d(x_1, x_2)} \right] + a_5 \left[\frac{d(x_1, x_2)(1 + d(x_0, x_1))}{1 + d(x_0, x_1)} \right] + a_6 \left[\frac{d(x_0, x_2)d(x_1, x_2)}{d(x_0, x_1) + d(x_1, x_2)} \right] \leq \\ &\leq a_1 d(x_0, x_1) + a_2 d(x_0, x_1) + a_2 d(x_1, x_2) + a_3 [d(x_0, x_1) + d(x_1, x_2)] + a_4 d(x_0, x_1) + a_5 d(x_1, x_2) + a_6 d(x_1, x_2) = \\ &= [a_1 + a_2 + a_3 + a_4]d(x_0, x_1) + [a_2 + a_3 + a_5 + a_6]d(x_1, x_2). \end{aligned}$$

Therefore, $[1 - (a_2 + a_3 + a_5 + a_6)]d(x_1, x_2) \leq [a_1 + a_2 + a_3 + a_4]d(x_0, x_1)$. This implies

$$d(x_1, x_2) \leq \left[\frac{a_1 + a_2 + a_3 + a_4}{1 - (a_2 + a_3 + a_5 + a_6)} \right] d(x_0, x_1).$$

Let

$$k = \left[\frac{a_1 + a_2 + a_3 + a_4}{1 - (a_2 + a_3 + a_5 + a_6)} \right].$$

Then, $d(x_1, x_2) \leq kd(x_0, x_1)$. Similarly,

$$d(x_2, x_3) \leq k^2 d(x_0, x_1), \quad d(x_3, x_4) \leq k^3 d(x_0, x_1).$$

Proceeding like this we get for any positive integer n , $d(x_n, x_{n+1}) \leq k^n d(x_0, x_1)$.

Now, if p is any positive integer,

$$\begin{aligned} d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{n+p-1}, x_{n+p}) \leq \\ &\leq k^n d(x_0, x_1) + k^{n+1} d(x_0, x_1) + \dots + k^{n+p-1} d(x_0, x_1) = \\ &= k^n (1 + k + \dots + k^{p-1}) d(x_0, x_1) \leq \\ &\leq \left[\frac{k^n}{1 - k} \right] d(x_0, x_1) \leq \left[\frac{k^n}{1 - k} \right] d(x_0, Tx_0). \end{aligned}$$

Since $a_1 + 2a_2 + 2a_3 + a_4 + a_5 + a_6 < 1$, then $0 \leq k < 1$. So, $d(x_n, x_{n+p}) \rightarrow 0$ as $n \rightarrow \infty$. Therefore, $\{x_n\}$ is a Cauchy sequence. As X is a complete quasi-metric space, we have

$$\lim_{n \rightarrow \infty} x_n = p \in X.$$

Since T is continuous, $\lim_{n \rightarrow \infty} Tx_n = Tp \Rightarrow \lim_{n \rightarrow \infty} Tx_{n+1} = Tp$. Thus,

$$Tp = p.$$

Hence, p is a fixed point of T .

Uniqueness: If $p \in X$ is a fixed point of T . Then by Equation (1)

$$\begin{aligned} d(p, p) &= d(Tp, Tp) \leq \\ &\leq a_1 d(p, p) + a_2 [d(p, Tp) + d(p, Tp)] + a_3 [d(p, Tp) + d(p, Tp)] + \\ &+ a_4 \left[\frac{d(p, p)d(p, Tp)}{d(p, p) + d(p, Tp)} \right] + a_5 \left[\frac{d(p, Tp)(1 + d(p, Tp))}{1 + d(p, p)} \right] + a_6 \left[\frac{d(p, Tp)d(p, Tp)}{d(p, p) + d(p, Tp)} \right] \leq \\ &\leq \left[a_1 + 2a_2 + 2a_3 + \frac{a_4}{2} + a_5 + \frac{a_6}{2} \right] d(p, p). \end{aligned}$$

Since $a_1 + 2a_2 + 2a_3 + \frac{a_4}{2} + a_5 + \frac{a_6}{2} < 1$ and $d(p, p) \geq 0$, we have $d(p, p) = 0$. Thus, $d(p, p) = 0$ if p is a fixed point of T . Suppose that p and q are two fixed points of T ($p \neq q$); that is, $p = Tp$ and $q = Tq$. We have

$$\begin{aligned} d(p, q) &= d(Tp, Tq) \leq \\ &\leq a_1 d(p, q) + a_2 [d(p, Tp) + d(q, Tq)] + a_3 [d(p, Tq) + d(q, Tp)] + \\ &+ a_4 \left[\frac{d(p, q)d(p, Tq)}{d(p, q) + d(q, Tq)} \right] + a_5 \left[\frac{d(q, Tq)(1 + d(p, Tp))}{1 + d(p, q)} \right] + a_6 \left[\frac{d(p, Tq)d(q, Tq)}{d(p, q) + d(q, Tq)} \right]. \end{aligned}$$

$$d(p, q) \leq [a_1 + a_3 + a_4]d(p, q) + a_3 d(q, p). \quad (2)$$

Similarly,

$$d(q, p) \leq [a_1 + a_3 + a_4]d(q, p) + a_3 d(p, q). \quad (3)$$

Subtracting equation (3) from (2), we get

$$|d(p, q) - d(q, p)| \leq |a_1 + a_4| |d(p, q) - d(q, p)|.$$

Since, $|a_1 + a_4| < 1$, we have $|d(p, q) - d(q, p)| = 0$. This implies

$$d(p, q) = d(q, p). \quad (4)$$

Putting equation (4) in (2), we get

$$d(p, q) \leq [a_1 + a_3 + a_4]d(p, q).$$

Since, $0 < a_1 + a_3 + a_4 < 1$, we have $d(p, q) = 0$. Similarly, $d(q, p) = 0$. Hence, $p = q$. $\square$

Example. Let $X = \mathbb{R}$ with q-metric defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ x^2 & \text{if } x \neq y \end{cases} \quad \forall x, y \in X$$

and let $T : X \rightarrow X$ such that $Tx = \frac{x}{2}$ be a continuous self-mapping. Suppose $a_1 = \frac{1}{3}$, $a_2 = \frac{1}{6}$, $a_3 = \frac{1}{12}$, $a_4 = \frac{1}{15}$, $a_5 = \frac{1}{20}$, $a_6 = \frac{1}{25}$, then T satisfies all the conditions of Theorem 1, and $x = 0$ is the unique fixed point of T in X .

Remark 1. In Theorem 1,

1. If $a_2 = a_3 = a_4 = a_5 = a_6 = 0$, then it extends the result of (Banach, 1922).
2. If $a_1 = a_2 = a_4 = a_5 = a_6 = 0$, then it extends the result of (Chatterjea, 1972).
3. If $a_1 = a_4 = a_5 = a_6 = 0$, then it extends the result of (Jojar et al., 2022).
4. If $a_1 = a_3 = a_4 = a_5 = a_6 = 0$, then it extends the result of (Kannan, 1968).
5. If $a_5 = 0$, then it extends the result of (Mittal et al., 2023).

Discussions and conclusion

In this paper, we proved a unique fixed point theorem for continuous self-mappings on complete quasi-metric spaces. The result is based on a generalized contraction condition involving several distance terms and extends classical fixed point theorems such as those by Banach, Kannan, and Chatterjea. Unlike many earlier results, our theorem does not require

additional structures like convexity or dislocation. An example was also given to support the main result. We hope this work will be useful for further studies in fixed point theory on asymmetric spaces.

Authors' contribution: Anurag – conceptualization, methodology, validation, formal analysis, original draft preparation; Aparna, Vyas – conceptualization, formal analysis, original draft preparation; Gibak, Kim – review and editing.

Sources of funding. This study did not receive any grant from a funding institution in the public, commercial, or non-commercial sectors.

References

- Ayoob, I., Chuan, N. Z., & Mlaiki, N. (2023). Quasi m-metric spaces. *AIMS Mathematics*, 8(5), 10228–10248. <https://doi.org/10.3934/math.2023518>
- Banach, S. (1922). Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundamenta Mathematicae*, 3, 133–181. <https://doi.org/10.4064/fm-3-1-133-181>
- Chatterjea, S. K. (1972). Fixed point theorem. *Comptes Rendus de l'Académie Bulgare des Sciences*, 25, 727–730.
- Jleli, M., & Samet, B. (2018). Fixed point theorem. *Comptes Rendus de l'Académie Bulgare des Sciences*, 9, 118–133. <https://doi.org/10.22436/jnsa.009.06.31>
- Jogar, G., Dolhare, U., Basude, S., Darkunde, N., & Swami, P. (2022). Some results on unique fixed point theorems in complete metric space. *International Journal of Mathematics Trends and Technology*, 68(6), 111–116. <https://doi.org/10.14445/22315373/IJMTT-V68I6P513>
- Kannan, R. (1968). Some results on fixed points. *Bulletin of the Calcutta Mathematical Society*, 60, 71–76.
- Mainik, A., & Mielke, A. (2005). Existence results for energetic models for rate-independent systems. *Calculus of Variations and Partial Differential Equations*, 22, 73–99. <https://doi.org/10.1007/s00526-004-0267-8>
- Mennucci, A. C. G. (2013). On asymmetric distances. *Analysis and Geometry in Metric Spaces*, 1, 200–231. <https://doi.org/10.2478/agms-2013-0004>
- Mielke, A., & Roubíček, T. (2003). A rate-independent model for inelastic behavior of shape memory alloys. *Multiscale Modeling & Simulation*, 1, 571–597. <https://doi.org/10.1137/S1540345903422860>
- Mittal, N., Kumar, P., Ansari, Z. K., & Vyas, A. (2023). Some fixed point theorems satisfying generalized contraction conditions in dislocated quasi metric space. *International Journal of Nonlinear Analysis and Applications*, 16(1), 1–7. <http://dx.doi.org/10.22075/ijnaa.2023.28150.3816>
- Rieger, M. O., & Zimmer, J. (2009). Young measure flow as a model for damage. *Zeitschrift für Angewandte Mathematik und Physik (ZAMP)*, 60, 1–32. <https://doi.org/10.1007/s00033-008-7016-3>
- Wilson, W. A. (1931). On quasi metric spaces. *American Journal of Mathematics*, 53, 675–684. <https://doi.org/10.2307/2371174>
- Woon, W. L., & Madnick, S. (2009). Asymmetric information distances for automated taxonomy construction. *Analysis and Geometry in Metric Spaces*, 21, 91–111. <https://doi.org/10.2139/ssrn.1256562>

Отримано редакцією журналу / Received: 05.08.25

Прорецензовано / Revised: 26.11.25

Схвалено до друку / Accepted: 16.04.26

Опубліковано / Published: 05.06.26

АНУРАГ, магістр
ORCID ID: 0009-0003-8391-7622
e-mail: anurag_kumar23@mru.ac.in
Університет Манава Рачна, Фаридабад, Індія

Апарна В'ЯС, д-р філософії, проф.
ORCID ID: 0000-0002-5301-8234
e-mail: aparna@mru.edu.in
Університет Манава Рачна, Фаридабад, Індія

Гібак КІМ, д-р філософії, проф.
ORCID ID: 0000-0001-5114-4117
e-mail: imkgb27@ssu.ac.kr
Університет Сунсіль, Сеул, Республіка Корея

РОЗШИРЕНА ТЕОРЕМА ПРО ЄДИНУ НЕРУХОМУ ТОЧКУ В ПОВНОМУ КВАЗІМЕТРИЧНОМУ ПРОСТОРІ

Основною метою дослідження є представлення нових теорем про нерухому точку в повних квазіметричних просторах. Теорія нерухомих точок є важливою галуззю, що поєднує теоретичну та прикладну математику. Хоча класичні результати, такі як принцип стискальних відображень Банаха, добре відомі для метричних просторів, у цій роботі зазначені концепції поширюються на більш загальний квазіметричний простір, що не задовольняє симетричну умову стандартного метричного простору. Ми доводимо теорему про єдину нерухому точку для неперервних відображень у повному квазіметричному просторі за нововизначеною, узагальненою умовою стискання. Цей результат суттєво узагальнює та розширює низку відомих теорем про нерухомі точки, згаданих у літературі. Для ілюстрації застосовності наших результатів наведено приклад. Наша робота робить внесок у розвиток теорії нерухомих точок в асиметричних просторах, які знаходять застосування у різних галузях, зокрема в дослідженні сплавів з ефектом пам'яті форми та риваннях Гамільтона - Якобі.

Ключові слова: нерухома точка, стискальне відображення, повний метричний простір, повний квазіметричний простір.

Автори заявляють про відсутність конфлікту інтересів. Спонсори не брали участі в розробленні дослідження; у зборі, аналізі чи інтерпретації даних; у написанні рукопису; в рішенні про публікацію результатів.

The authors declare no conflicts of interest. The funders had no role in the design of the study; in the collection, analyses or interpretation of data; in the writing of the manuscript; in the decision to publish the results.