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BOUNDS SIMULATION FOR FLUID FLOW MODEL OF THE SECOND ORDER

This paper presents an approximation formula for the conditional cumulative distribution function of the maximum of a Brownian bridge with a given minimum. The formula enables us to develop an algorithm that generates the minimum and the upper bound of the maximum for a Brownian bridge trajectory. Thus, we can determine the simulation procedure for the upper and lower limits of a second-order fluid flow model. The demonstration of the results of the algorithm for a specific set of parameters is given. The extrema empirical distributions obtained using the sequential discretisation method and the approximate quantile method are compared. The possibility of sampling points in the middle of intervals, as well as choosing one of the subintervals at which the minimum is achieved, is considered.

Key words: strong simulation, acceptance-rejection sampling with squeezing, quantile transformation.

AMS 2020 classification: 60G51, 65C05, 91G60.

Introduction

Fluid models can be considered as a special class of Markov additive processes that allow the study of inhomogeneous dynamics. The parameters are assumed to depend on the state of the external environment, which can be described in terms of a finite Markov chain. Two main types of fluid models are recognized: first- and second-order. The first-order models imply piecewise linear paths that change direction at the transition moments of the environment, or regime-switching moments. A detailed list of references on the applications of these models in various fields, as well as an overview of known extensions, can be found in (Latouche, & Nguyen, 2018). Such models are appropriate if the system characteristics can be completely determined by first moments. The second-order model emerges when controlling the first and second moments of the rates of change is necessary (cf. (Karandikar, & Kulkarni, 1995)). This model evolve like a Brownian motion, but the drift and volatility depend on the environment state. It is also referred to as Markov-modulated Brownian motion (see, for example, (Latouche, 2025)).

Fluid models are versatile enough to represent a variety of phenomena while remaining analytically tractable. A matrix-analytical framework has been developed for them. Within this framework, many probabilistic characteristics (at least in terms of some integral transform like the Laplace transform) can be expressed in explicit form as a function of solutions to a certain quadratic matrix equation (see, for example, (Ahn, & Meini, 2019)). The framework stays relevant for some extensions like models with phase-type jumps (see, e.g., (Breuer, 2010a; D'Auria et al., 2010)), models with barriers ((Ivanovs, 2010; Breuer, 2010b; Horváth, & Telek, 2017), to name just a few), and with the level dependent evolution (see (Almoussa, Horváth, & Telek, 2020) and the references therein).

Including jumps in the model whose distribution differs from the phase-type distribution (or, more generally, from a distribution with a rational characteristic function) significantly complicates the possibility of obtaining an explicit solution. The problem of establishing effective numerical procedures becomes apparent in this case. One direct approach is to use the Monte Carlo method. Rather than trying to derive analytical expressions for probabilistic characteristics, we can generate a large enough number of paths to get the empirical distribution of the characteristic of interest. Under the assumption that the jump part is independent of the diffusion part, the main issue will be related to the latter (sample paths of it are infinite dimensional random elements). For a general overview of the issue see, e.g., (Pollock, 2013). Using a discretisation scheme (simulating from the finite dimensional distributions) introduces the bias to the Monte Carlo estimator. To overcome this issue, (Beskos, Peluchetti, & Roberts, 2012) presented the ε -strong algorithm, a rejection sampling method for constructing upper and lower Brownian path boundaries. This method allows for unbiased Monte Carlo estimators over path expectations. For an extension to the case of jump diffusion, see (Pollock, Johansen, & Roberts, 2016). For some alternative approaches, see, for instance, (Gribaudo, Iacono, & Manini, 2021), (D'Onofrio, & Lanteri, 2023), (Herrmann, & Massin, 2023), and the references therein.

The ε -strong algorithm involves iterative use of the rejection method at all steps, including the initialization one. At this step, it was proposed to select from an increasing sequence of nested space intervals the one (the so-called Bessel layer) that contains the entire trajectory at a given time interval (for more details, see (Pollock, Johansen, & Roberts, 2016, Section 7.1)). In this paper, we present and justify an algorithm for modeling upper and lower bounds using the inversion method. Specifically, we propose to simulate the endpoints of a path over each time interval with the minimum and an upper bound of the maximum (cf. (Pollock, Johansen, & Roberts, 2016, Section 7.2)). The possibility of midpoint values simulation with subsequent inclusion of refinement and bisection steps of the ε -strong algorithm for certain subintervals is also discussed. This can provide enhanced control over path behavior on the corresponding interval. Thus, the proposed algorithm complements the existing range of diffusion modeling methods.

This paper is structured as follows. In Section 1, we provide the definition of the second-order fluid model. Section 2 contains an in-depth discussion of the modeling methods employed. In Section 3, we present a monotonic sequence that approximates the conditional cdf of the Brownian bridge maximum with given minimum from below. We use this sequence to justify the algorithm for modeling the minimum and an upper bound of the maximum. An example for comparison with the results obtained by the standard discretization method is provided. Section 4 is devoted to discussing the possibility

of modeling midpoint values in order to clarify the behavior of the trajectory on subintervals. Finally, Section 5 describes the simulation procedure that combines the steps of constructing a path envelope with endpoints and midpoints, taking regime-switching into account.

1. The model

Let $J = \{J_t, t \geq 0\}$ be a homogeneous irreducible Markov chain with right continuous sample paths and finite state space $E = \{1, \dots, N\}$. Denote by $Q = \|Q_{ij}\|_{i,j=1}^N$ the rate transition matrix and by $\pi = (\pi_1 \dots \pi_N)$ the invariant distribution. For certainty, we assume that π is also the initial distribution of the chain. J is usually referred to as a phase or regime process, and J_t can be interpreted as the environment state at the moment t . The environment controls the flow parameters: $a_i \in \mathbb{R}$ (drift) and $\sigma_i^2 \geq 0$ (volatility), $i \in E$. The model will be of the second order if $\exists i \in E: \sigma_i^2 > 0$, and if $\forall i \in E: \sigma_i^2 = 0$, we get the first order model.

Let $W = \{W(t), t \geq 0\}$ be a Wiener process. We assume that J and W are mutually independent. The fluid level process is defined as

$$X_t = X_0 + \int_0^t a_{J_s} ds + \int_0^t \sigma_{J_s}^2 dW_s,$$

where X_0 is the initial level. That is, X follows a Brownian motion in between any two jumps of J . The process $\{X, J\}$ is called a fluid flow process, and it is a special case of Markov additive processes (see, for example, (Çinlar, 1972)).

The simulation of J is straightforward. We begin with a randomly selected initial state according to the initial distribution, say $J_0 = j$. The time J spends in the state is exponentially distributed with rate $-Q_{jj}$. Then, sample the next state from the distribution $\frac{Q_{ji}}{-Q_{jj}}$, $i \neq j$, and generate independently the next inter-transition time, etc.

In between transitions, X is a pure diffusion. We can split the time interval, say, into 2^m parts, $m \in \mathbb{N}$, and simulate the sample path at each point of partition iteratively. This discretization scheme does not sufficiently characterize the simulated sample paths. A path's entry into a region cannot be accurately determined simply by examining its position at a limited number of points. To gain more control over the path behavior within the partition interval, it can be helpful to get upper and lower bounds. Since we fix the initial and terminal values, the Brownian motion is actually a Brownian bridge. Although there is an analytical expression for the joint distribution of the minimum and maximum, it is represented in terms of an infinite series, which complicates the simulation. In the next section, we consider some versions of two basic techniques to simulate a random variable, namely, quantile transformation and the accept-reject method (see, e.g., (Brandimarte, 2014, Chapter 5) or (Owen, 2013, Chapter 4)), which can be useful for overcoming complications.

2. Simulation methods

Let $\tilde{g}(x)$ be a nonnegative function with $\int_{-\infty}^{\infty} \tilde{g}(x) dx = c_g < \infty$. Assume that we can simulate a random variable $\zeta \sim g(x) = \frac{1}{c_g} \tilde{g}(x)$, but we need to simulate a random variable $\xi \sim f(x)$ with

$$f(x) \propto p(x) \tilde{g}(x),$$

where $p(x)$ is the probability of an event which we only know up to the alternate sequence:

$$p_2(x) \leq p_4(x) \leq \dots \leq p_{2n-2}(x) \leq p_{2n}(x) \leq \dots \leq p(x) \leq \dots \leq p_{2n+1}(x) \leq p_{2n-1}(x) \leq \dots \leq p_1(x),$$

$$\lim_{n \rightarrow \infty} p_{2n}(x) = p(x) = \lim_{n \rightarrow \infty} p_{2n-1}(x), \quad \forall x \in \mathbb{R}.$$

Assume $p_{-1}(x) \equiv 1$ and $p_0(x) \equiv 0$. In this case we can use the following accept-reject algorithm with squeezing (see, for instance, (Pollock, 2013, Algorithm 2.6.1)).

1. Set $n = 1$. Simulate independently $\eta \sim Uniform(0, 1)$ and $\zeta \sim g(x)$.
2. If $\eta > p_n(\zeta)$, then back to Step 1 (reject ζ). If not, go further.
3. If $\eta < p_{n+1}(\zeta)$, then put $\xi = \zeta$ and stop (accept ζ). If not, increase n by 1 and go to Step 2.

Proposition 1. Random variable ξ obtained by the accept-reject algorithm with squeezing, has probability density function f .

Let ν be the number of steps we need to simulate a sample from f of size one. Then for $n \geq 2$

$$\Pr\{\nu = n\} = \sum_{\substack{q, k_1, \dots, k_{q+1} \in \mathbb{N}: \\ q \leq n, n - q \text{ is even,} \\ \sum_{i=1}^{q+1} k_i = (n - q)/2}} \prod_{i=1}^q R_{2k_i-1} A_{2k_{q+1}}, \quad (1)$$

where

$$\begin{aligned} R_{2k-1} &= \frac{1}{c_g} \int_{-\infty}^{\infty} (p_{2k-3}(z) \wedge 1 - p_{2k-1}(z) \wedge 1) \tilde{g}(z) dz, \\ A_{2k} &= \frac{1}{c_g} \int_{-\infty}^{\infty} (p_{2k}(z) \vee 0 - p_{2k-2}(z) \vee 0) \tilde{g}(z) dz, \\ x \vee y &:= \max\{x, y\}, \quad x \wedge y := \min\{x, y\}. \end{aligned}$$

Proof. Denote $\tilde{f}(x) = p(x)\tilde{g}(x)$ and $c_f = \int_{-\infty}^{\infty} \tilde{f}(x) dx$. Using the total probability law and independence of η and ζ we get

$$\begin{aligned} \Pr\{\xi < x\} &= \Pr\{\xi < x | \zeta \text{ is accepted}\} \Pr\{\zeta \text{ is accepted}\} + \Pr\{\xi < x | \zeta \text{ is rejected}\} \Pr\{\zeta \text{ is rejected}\} = \\ &= \Pr\{\xi < x | \zeta \text{ is accepted}\} \Pr\{\zeta \text{ is accepted}\} + \Pr\{\xi < x\} \Pr\{\zeta \text{ is rejected}\} = \\ &= \Pr\{\zeta < x, \zeta \text{ is accepted}\} + \Pr\{\xi < x\} \Pr\{\zeta \text{ is rejected}\}. \end{aligned} \quad (2)$$

From the algorithm

$$\begin{aligned} \Pr\{\zeta \text{ is rejected}\} &= \Pr\{\exists k \in \mathbb{N} : \eta > p_{2k-1}(\zeta)\} = \Pr\{\eta > p(\zeta)\} = \Pr\left\{\eta > \frac{\tilde{f}(\zeta)}{\tilde{g}(\zeta)}\right\} = \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{\{\eta > \frac{\tilde{f}(\zeta)}{\tilde{g}(\zeta)}\}} \mid \zeta\right]\right] = \\ &= \mathbb{E}\left[1 - \frac{\tilde{f}(\zeta)}{\tilde{g}(\zeta)}\right] = 1 - \int_{-\infty}^{\infty} \frac{\tilde{f}(z)}{\tilde{g}(z)} g(z) dz = 1 - \frac{c_f}{c_g} \end{aligned}$$

and

$$\Pr\{\zeta < x, \zeta \text{ is accepted}\} = \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{\{\zeta < x\}} \mathbb{1}_{\{\eta < \frac{\tilde{f}(\zeta)}{\tilde{g}(\zeta)}\}} \mid \zeta\right]\right] = \mathbb{E}\left[\mathbb{1}_{\{\zeta < x\}} \frac{\tilde{f}(\zeta)}{\tilde{g}(\zeta)}\right] = \int_{-\infty}^x \frac{\tilde{f}(z)}{\tilde{g}(z)} g(z) dz = \frac{c_f}{c_g} \int_{-\infty}^x f(z) dz.$$

Substitution in (2) yields $\Pr\{\xi < x\} = \int_{-\infty}^x f(z) dz$. Thus, $\xi \sim f(x)$.

To get (1) just note that

$$\begin{aligned} A_{2k} &= \Pr\{\zeta \text{ is accepted in } 2k \text{ steps}\} = \Pr\{p_{2k-2}(\zeta) < \eta < p_{2k}(\zeta)\}, \\ R_{2k-1} &= \Pr\{\zeta \text{ is rejected in } 2k-1 \text{ steps}\} = \Pr\{p_{2k-1}(\zeta) < \eta < p_{2k-3}(\zeta)\}. \end{aligned}$$

□

From a practical point of view, the use of formula (1) to estimate the complexity of the algorithm may require considerable analytical effort. In accordance with the algorithm, we will check within the loop whether a proposal falls within the acceptance or rejection region. One way to control the execution time of the loop is to impose a maximum number of steps. Consequently, it would be suitable to consider the probability that the check of the proposal will require no more steps than the pre-established limit.

Fix $n_{max} \in \mathbb{N}$ and denote the events

$$H = \{\text{Check of the proposal takes more than } n_{max} \text{ steps}\}, S = \bar{H},$$

$$A = \{\text{The proposal is accepted}\}, R = \bar{A}.$$

For concise notation, we will use P_E as the probability of event E and P_{AB} as $\Pr(A \cap B)$.

Proposition 2. Let $\tilde{\nu}$ be the number of proposal checks that take more than n_{max} steps. Then

$$\Pr\{\tilde{\nu} = 0\} = \frac{P_{AS}}{1 - P_{RS}}, \Pr\{\tilde{\nu} = m\} = \left(\frac{P_{RH}}{1 - P_{RS}}\right)^{m-1} \frac{P_A P_H}{(1 - P_{RS})^2}, m \in \mathbb{N}. \quad (3)$$

Proof. Event $\{\tilde{\nu} = 0\}$ corresponds to the case when all checks take less than or equal to n_{max} steps. So,

$$\Pr\{\tilde{\nu} = 0\} = P_{AS} \sum_{k=0}^{\infty} P_{RS}^k = \frac{P_{AS}}{1 - P_{RS}}.$$

Conditioning on whether the last step of acceptance take more than n_{max} steps and on the number of preceding rejections, we have

$$\Pr\{\tilde{\nu} = m\} = P_{AH} \sum_{k=m-1}^{\infty} C_k^{m-1} P_{RS}^{k-m+1} P_{RH}^{m-1} + P_{AS} \sum_{k=m}^{\infty} C_k^m P_{RS}^{k-m} P_{RH}^m.$$

Since $\frac{1}{(1 - P_{RS})^{m+1}} = \sum_{k=m}^{\infty} C_k^m P_{RS}^{k-m}$ we get

$$\Pr\{\tilde{\nu} = m\} = \frac{P_{AH} P_{RH}^{m-1}}{(1 - P_{RS})^m} + \frac{P_{AS} P_{RH}^m}{(1 - P_{RS})^{m+1}} = \left(\frac{P_{RH}}{1 - P_{RS}}\right)^{m-1} \left(\frac{P_{AH} (1 - P_{RS}) + P_{AS} P_{RH}}{(1 - P_{RS})^2}\right).$$

The observation that $P_{AH} (1 - P_{RS}) + P_{AS} P_{RH} = P_A P_H$ completes the proof.

□

Remark 1. From Proposition 2 we have that the distribution of $\tilde{\nu}$ is the mixture (with parameter $\frac{P_{AS}}{1 - P_{RS}}$) of the distribution concentrated at 0 and the geometric distribution $Geom\left(1 - \frac{P_{RH}}{1 - P_{RS}}\right)$. This is to be expected as we wait for a particular event

to occur in successive independent experiments and count only those that meet the given criteria. From (3) we deduce

$$E \tilde{\nu} = \left(\frac{1}{1 - \frac{P_{RH}}{1 - P_{RS}}} \right)^{-2} \frac{P_A P_H}{(1 - P_{RS})^2} = P_H \frac{1}{P_A}.$$

This result can be obtained as a consequence of Wald's equality and the representation $\tilde{\nu} = \sum_{i=1}^{\kappa} X_i$, where κ is the number of proposals we need to generate $\xi \sim f(x)$ and $\{X_i\}_{i=1}^{\infty}$ are i.i.d. indicators of event A . From the proof of Proposition 1 the probability of acceptance is $\frac{c_f}{c_g}$. The number of proposals that we must make before one is accepted has the distribution $Geom\left(\frac{c_f}{c_g}\right)$ and the expected number of proposals is $\frac{c_g}{c_f}$. We count only the proposals that require complex verification which can happen with the probability

$$P_H = \frac{1}{c_g} \int_{-\infty}^{\infty} d_{n_{max}}(z) \tilde{g}(z) dz,$$

where

$$d_n(z) = \begin{cases} \mathbb{I}_{\{p_n(z) > 1\}} + p_n(z) \mathbb{I}_{\{p_n(z) < 1\}} - p_{n-1}(z) \mathbb{I}_{\{p_{n-1}(z) > 0\}}, & n \text{ is odd,} \\ \mathbb{I}_{\{p_{n-1}(z) > 1\}} + p_{n-1}(z) \mathbb{I}_{\{p_{n-1}(z) < 1\}} - p_n(z) \mathbb{I}_{\{p_n(z) > 0\}}, & n \text{ is even.} \end{cases}$$

Thus, the expected number of such proposals is

$$E \tilde{\nu} = \frac{\int_{-\infty}^{\infty} d_{n_{max}}(z) \tilde{g}(z) dz}{\int_{-\infty}^{\infty} p(z) \tilde{g}(z) dz}.$$

Assuming that n_{max} is even and large enough (such that $p_{n_{max}}(z) \geq 0$ and $p_{n_{max}-1}(z) \leq 1$) we get the upper estimate

$$E \tilde{\nu} \leq \frac{\int_{-\infty}^{\infty} p_{n_{max}-1}(z) \tilde{g}(z) dz}{\int_{-\infty}^{\infty} p_{n_{max}}(z) \tilde{g}(z) dz} - 1. \quad (4)$$

Now, consider the following modification of the quantile transformation method. Let ξ be a random variable with strictly monotonic continuous cdf $F(x) = \Pr\{\xi \leq x\}$. The exact formula for $F(x)$ is not known. We can only approximate it from below by $F_n(x)$: $F_n(x) \leq F(x)$, $F_n(x)$ is strictly monotone and continuous on the set of nonnegative values of F_n , and $\lim_{n \rightarrow \infty} F_n(x) = F(x)$, $\forall x \in \mathbb{R}$. Define $\tilde{F}_n(x) = F_n(x) \mathbb{I}_{\{F_n(x) \geq 0\}}(x)$ and $\tilde{F}_n^{-1}(x) = \inf\{y : \tilde{F}_n(y) \geq x\}$, then $\forall x \in \mathbb{R}$: $\tilde{F}_n(x) \uparrow F(x)$ and $\forall y > 0$: $\tilde{F}_n^{-1}(y) \downarrow F^{-1}(y)$ (see Fig. 1). Therefore, the following statement holds.

Lemma 1. Let $\eta \sim Uniform(0, 1)$ and put $\xi_n = \tilde{F}_n^{-1}(\eta)$. Then the sequence $\{\xi_n\}_{n=1}^{\infty}$ converges almost surely to a random variable with cdf F .

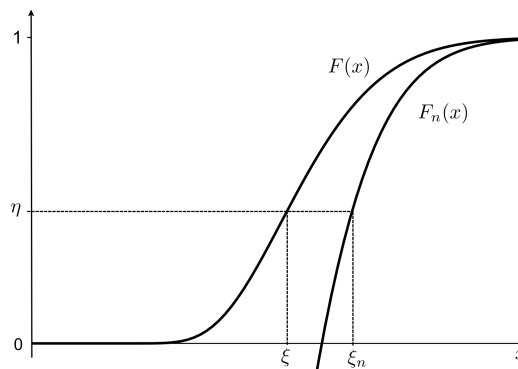


Fig. 1. Upper bound simulation for ξ

Lemma 1 justifies the quantile method of modeling an upper bound that monotonically approaches a random variable ξ in the case when its cumulative distribution function is known to within an approximating sequence. The exact representation for $\tilde{F}_n^{-1}(x)$ is the exception rather than the rule. To find $\xi_n = \tilde{F}_n^{-1}(\eta)$, we will use a numeric procedure for solving the nonlinear equation (e.g., the van Wijngaarden-Dekker-Brent method (Forsythe, Malcolm, & Moler, 1977, Section 7.2)). In this case, we need to be careful that we are trying to approach the solution from above.

3. Simulation of the minimum and an upper bound of the maximum for a Brownian bridge

Let $\{B_{x,l,y}(t), t \in [0, l]\}$ be the Brownian bridge process with value $x \in \mathbb{R}$ at 0 and value $y \in \mathbb{R}$ at $l > 0$. Since

$$B_{x,l,y}(t) \sim x + \frac{t}{l}(y - x) + \frac{l-t}{\sqrt{l}} W_{\frac{t}{l}}, 0 \leq t \leq l,$$

we have that the event that the Brownian bridge stays in an interval corresponds to the event that a Wiener process does not cross linear boundaries (for more details see (Casella, & Roberts, 2008, Appendix 1)):

$$\Pr\{u < B_{x,l,y}(t) < v, \forall t \in [0, l]\} = \Pr\left\{-\frac{(y-u)t + (x-u)}{\sqrt{l}} < W_t < \frac{(v-y)t + (v-x)}{\sqrt{l}}, \forall t \in [0, l]\right\}.$$

Set $B_{x,l,y}^- = \inf_{t \in [0, l]} B_{x,l,y}(t)$ and $B_{x,l,y}^+ = \sup_{t \in [0, l]} B_{x,l,y}(t)$. Then from (Beskos, Peluchetti, & Roberts, 2012, (4.3)) we get

$$\Pr\{B_{x,l,y}^- > u, B_{x,l,y}^+ < v\} = \left(1 - \sum_{n=1}^{\infty} H_n(l, v-u, x-u, y-u)\right) \mathbb{I}_{\{u < x, y < v\}}, \quad (5)$$

where

$$H_n(l, \Delta, \Delta_x, \Delta_y) = e^{-q_1^+} + e^{-q_2^+} - e^{-q_1^-} - e^{-q_2^-}, \quad \Delta \geq \Delta_x, \Delta_y \geq 0,$$

with

$$q_1^+ = \frac{2}{l}(n\Delta - \Delta_x)(n\Delta - \Delta_y), q_2^+ = \frac{2}{l}((n-1)\Delta + \Delta_x)((n-1)\Delta + \Delta_y),$$

$$q_1^- = \frac{2n\Delta}{l}(n\Delta + (\Delta_x - \Delta_y)), q_2^- = \frac{2n\Delta}{l}(n\Delta + (\Delta_y - \Delta_x)).$$

As a consequence, we can get the marginal cdfs (see also (Doob, 1949, (4.2)))

$$F_{B_{x,l,y}^-}(u) = \begin{cases} e^{-\frac{2}{l}(x-u)(y-u)}, & u \leq x, y, \\ 1 & \text{otherwise,} \end{cases} \quad F_{B_{x,l,y}^+}(u) = \begin{cases} 1 - e^{-\frac{2}{l}(v-x)(v-y)}, & v \geq x, y, \\ 0 & \text{otherwise,} \end{cases}$$

and expectations

$$\mathbb{E} B_{x,l,y}^- = x \wedge y - \sqrt{\frac{\pi l}{8}} e^{\frac{(x-y)^2}{2l}} \operatorname{erfc}\left(\frac{x \vee y - x \wedge y}{\sqrt{2l}}\right),$$

$$\mathbb{E} B_{x,l,y}^+ = x \vee y + \sqrt{\frac{\pi l}{8}} e^{\frac{(x-y)^2}{2l}} \operatorname{erfc}\left(\frac{x \vee y - x \wedge y}{\sqrt{2l}}\right),$$

where $\operatorname{erfc}(x) = 1 - \sqrt{\frac{2}{\pi}} \int_0^x e^{-t^2} dt$ is the complementary error function. Each addend of $H_n(l, v-u, x-u, y-u)$ is monotone in u , and from (Jones, 2001, Ch. 16, Sec C) we can use term-wise differentiation of the series in (4), which gives the following statement.

Proposition 3. *The conditional cdf of the Brownian bridge maximum with given minimum $u \leq x$ can be represented as the series*

$$F_{B_{x,l,y}^+ | B_{x,l,y}^- = u}(v) = \begin{cases} \frac{1}{f_{B_{x,l,y}^-}(u)} \sum_{n=1}^{\infty} G_n(l, v-u, x-u, y-u), & v \geq x \vee y, \\ 0 & v < x \vee y, \end{cases}$$

where $f_{B_{x,l,y}^-}(u)$ is the pdf of minimum, and

$$G_n(l, \Delta, \Delta_x, \Delta_y) = e^{-q_1^+} p_1^+ + e^{-q_2^+} p_2^+ - e^{-q_1^-} p_1^- - e^{-q_2^-} p_2^-, \quad \Delta \geq \Delta_x, \Delta_y \geq 0,$$

with

$$p_1^+ = \frac{2(n-1)}{l}(2n\Delta - \Delta_x - \Delta_y), p_2^+ = \frac{2n}{l}(2(n-1)\Delta + \Delta_x + \Delta_y),$$

$$p_1^- = \frac{2n}{l}(2n\Delta + \Delta_x - \Delta_y), p_2^- = \frac{2n}{l}(2n\Delta + \Delta_y - \Delta_x).$$

Theorem 1. *Function $\tilde{F}_m(v) = \tilde{F}_m(l, x, y, u, v) = \frac{1}{f_{B_{x,l,y}^-}(u)} \sum_{n=1}^m G_n(l, v-u, x-u, y-u) \mathbb{I}_{\{v \geq x \vee y\}}$ is continuous and strictly monotone on the set on which it is non-negative. Moreover, $\tilde{F}_m(v) \uparrow F_{B_{x,l,y}^+ | B_{x,l,y}^- = u}(v)$ as $m \rightarrow \infty$.*

Proof. Fix $n \geq 2, l > 0, \Delta_x \geq 0, \Delta_y \geq 0$, then the function G_n can be considered as a linear combination of functions of the following type $f(\Delta) = e^{-q(\Delta)} p(\Delta)$ with $q(\Delta) = (a\Delta + b_1)(a\Delta + b_2)$, $p(\Delta) = k\Delta + m$, $a > 0, k > 0, \Delta \geq \tilde{\Delta}$, where $\tilde{\Delta} := \max\{0, -\frac{b_1}{a}, -\frac{b_2}{a}, -\frac{m}{k}\}$. For the derivative, we have the representation

$$f'(\Delta) = e^{-q(\Delta)} (-q'(\Delta) p(\Delta) + p'(\Delta)) = e^{-q(\Delta)} (-a(a\Delta + b_1 + a\Delta + b_2)(k\Delta + m) + k).$$

Since $0 \leq (a\Delta + b_1 + a\Delta + b_2)(k\Delta + m) \uparrow +\infty$ as $\Delta \uparrow +\infty$, we have only the two possible cases for behavior of f . If $f'(\tilde{\Delta}) > 0$, then f first increases to some point $\Delta^* > \tilde{\Delta}$ (the corresponding values of Δ^* for each term of G_n are $\frac{\sqrt{l} + \Delta_x + \Delta_y}{2n}$, $\frac{\sqrt{l} - \Delta_x - \Delta_y}{2(n-1)}$, $\frac{\sqrt{l} - \Delta_x + \Delta_y}{2n}$, and $\frac{\sqrt{l} + \Delta_x - \Delta_y}{2n}$), and then monotonically approaches 0. If $f'(\tilde{\Delta}) \leq 0$, then f monotonically approaches 0 from the starting point (see Fig. 2).

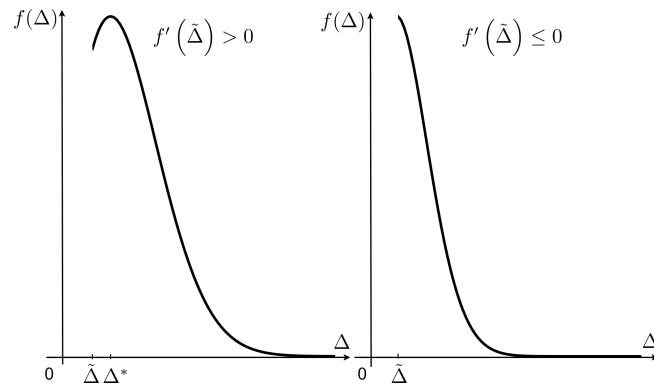


Fig. 2. Cases of possible behavior of $e^{-q(\Delta)}p(\Delta)$

Since $q_1^+ \leq q_1^-$ and $q_2^+ \leq q_2^-$, $e^{-q_1^+}p_1^+ + e^{-q_2^+}p_2^+$ tends to 0 more slowly than $e^{-q_1^-}p_1^- + e^{-q_2^-}p_2^-$. Hence, G_n starting from some $\bar{\Delta}_n$ will be positive monotonically decreasing to 0 (see the top right graph in Fig. 3), and this $\bar{\Delta}_n$ will decrease monotonically in n (the bottom left graph in Fig. 3).

If $n = 1$, then $e^{-q_1^-}p_1^-$ and $e^{-q_2^-}p_2^-$ show the similar behavior as shown in Fig. 2. While $e^{-q_1^+}p_1^+$ and $e^{-q_2^+}p_2^+$ are constant in Δ . More precisely, $e^{-q_1^+}p_1^+ + e^{-q_2^+}p_2^+|_{n=1, \Delta=v-u, \Delta_x=x-u, \Delta_y=y-u} = f_{B_{x,l,y}^-}(u)$. The term G_1 as function of Δ starts from the non-positive value (for certainty, let $\Delta_x \leq \Delta_y$, then $G_1(l, \Delta_y, \Delta_x, \Delta_y) = -e^{-\frac{2\Delta_y}{l}(2\Delta_y - \Delta_x)} \frac{2}{l} (3\Delta_y - \Delta_x) \leq 0$), reaches zero at some point $\bar{\Delta}_1$, and then monotonically approaches $f_{B_{x,l,y}^-}(u)$. So, the sum $\sum_{i=1}^n G_i$ is continuous and, starting from some point, strictly monotone in Δ (the bottom right graph in Fig. 3).

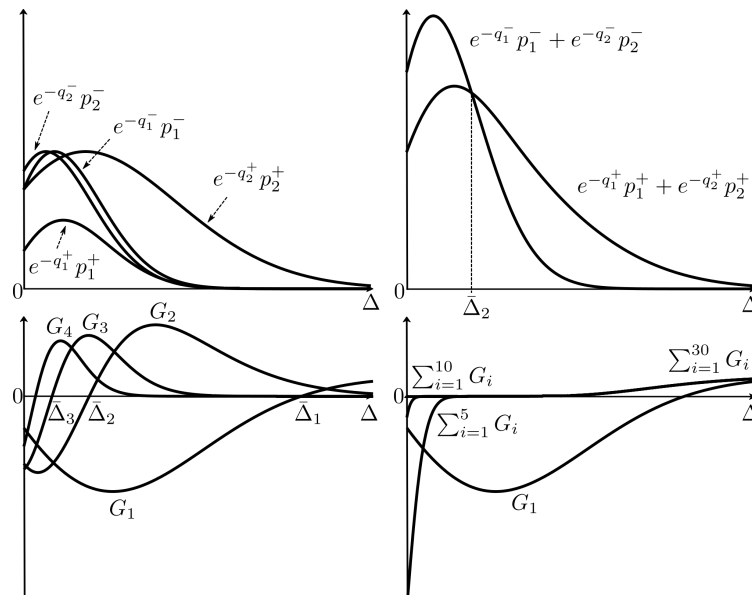


Fig. 3. Behavior of $\sum_{i=1}^m G_i$ with $l = 2, \Delta_x = 0.1, \Delta_y = 0.2$

□

Combining Lemma 1 with Theorem 1 gives us the following modified quantile transformation method for modeling the minimum and an upper bound of the maximum. Let $\tilde{F}_m^{-1}$ be the inverse function of $\tilde{F}_m$ restricted to the set $\{v : \tilde{F}_m(v) \geq 0\}$ and ζ_1, ζ_2 are independent $Unifom(0, 1)$ -distributed random variables. Define $\xi = F_{B_{x,l,y}^-}^{-1}(\zeta_1)$ and $\eta_m = \tilde{F}_m^{-1}(\zeta_2)|_{u=\xi}$, then $\eta_m \geq \eta$ and

$$\Pr\{\xi < r, \eta_m < q\} = E \mathbb{I}_{\left\{\zeta_1 < F_{B_{x,l,y}^-}^{-1}(r)\right\}} \tilde{F}_m(q)|_{u=F_{B_{x,l,y}^-}^{-1}(\zeta_1)} = \int_{-\infty}^r \tilde{F}_m(q)|_{u=z} f_{B_{x,l,y}^-}(z) dz.$$

By Lebesgue's dominated convergence theorem

$$\Pr\{\xi < r, \eta_m < q\} \xrightarrow{m \rightarrow \infty} \int_{-\infty}^r F_{B_{x,l,y}^+|B_{x,l,y}^-}^{-1}(q) f_{B_{x,l,y}^-}(z) dz = \Pr\{B_{x,l,y}^- < r, B_{x,l,y}^+ < q\}.$$

Example. Just to illustrate, we fixed $x = 2.1$, $y = 2.2$, $l = 1$, and simulated 100000 of Brownian bridge trajectories using standard discrete-time approximation. Namely, we simulate a Wiener process path $\{W_{t_i}\}_{i=1}^{2^m}$ at points $t_i = \frac{il}{2^m}$, $m = 12$, and calculate $B_{x,l,y}(t_i) = x + \frac{t_i}{l}(y - x) + W_{t_i} - \frac{t_i}{l}W_l$, $i = \overline{1, m}$. The points $(\min_{1 \leq i \leq 2^m} B_{x,l,y}(t_i), \max_{1 \leq i \leq 2^m} B_{x,l,y}(t_i))$ determined for each trajectory are presented in Fig. 4 (on the left). The contours in Fig. 4 are constructed based on the approximation of the joint density of the minimum and maximum, obtained as the sum of the derivatives with respect to u and v of the first 30 terms in (4). The graph on the right in Fig. 4 shows the results of 100000 simulations using the modified quantile transformation. To find $\tilde{F}_m^{-1}(\zeta_2)|_{u=\xi}$, we used the van Wijngaarden-Dekker-Brent method, iteratively increasing m to no more than 50, until the difference between successive solutions became less than 10^{-8} . The sample and theoretical values of the expectations, together with the 0.025 and 0.975 quantiles, rounded to 6 significant figures, are shown in Tab. 1.

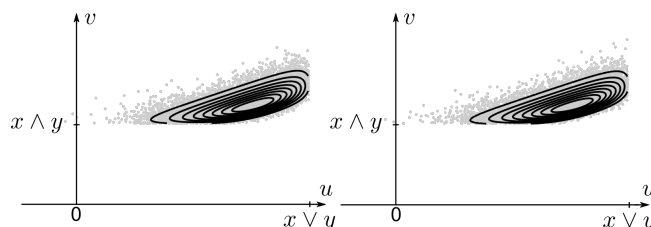


Fig. 4. Generated minimum and maximum of the Brownian bridge using sequential discrete-time simulation (left) and the approximate quantile method (right)

Table 1

Estimates and exact values of the mean and quantiles for extrema				
	Minimum $B_{x,l,y}^-$		Maximum $B_{x,l,y}^+$	
Theoretical Mean	1.52037		2.77963	
Theoretical Quantiles	(0.790978, 2.02688)		(2.27312, 3.50902)	
	Discrete-time approximation	Quantile transform approximation	Discrete-time approximation	Quantile transform approximation
Sample Mean	1.52782	1.52104	2.76925	2.78048
Sample Quantiles	(0.795669, 2.03636)	(0.799451, 2.02684)	(2.26324, 3.50005)	(2.2754, 3.50600)

In general, the discrete-time approximation gives us an overestimate for the minimum and underestimate for the maximum. To demonstrate, we simulated 1000 Brownian bridge paths at points $\{t_i\}_{i=1}^{2^m}$, found the average values for the minimum and maximum, and repeated this process 150 times. Similar steps were performed for the quantile transformation. The smooth kernel density estimates for the resulting data are shown in Fig. 5.

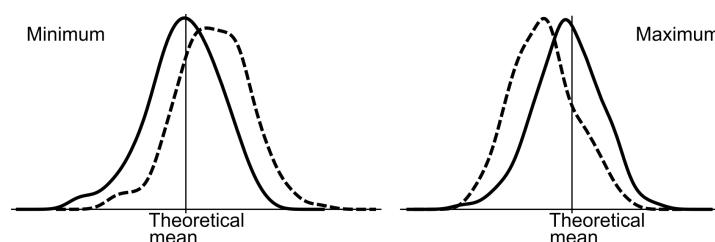


Fig. 5. Kernel-smoothed empirical marginal distributions for the average of minimum (left) and maximum (right) using sequential discrete-time simulation (dashed lines) and the approximate quantile method (solid lines)

Remark 2. It is possible to apply the considered algorithm to model Brownian motion with drift a and volatility σ^2 . To do so, substitute $l\sigma^2$ instead of l in the obtained relations (see, e.g. (Metwally, & Atiya, 2002, Section II)). According to Girsanov's theorem, the value of the drift is not essential (Beskos, Peluchetti, & Roberts, 2012, Section 6.1).

Example. Let $a = -1$, $\sigma = \sqrt{5}$. We split the interval $[0, 1]$ into 80 subintervals of the same length and start from $x = 2$. Fig. 6 shows sampled points of a Brownian motion path with corresponding lower and upper bounds.

The first passage time to the level 0 has an inverse Gaussian distribution (see, e.g., (Karatzas, & Shreve, 1998, Ch. 3, (5.12))) and the probability of crossing this level on the interval $[0, 1]$ up to 6 digits is 0.527339. We did a series of 1000 runs. In each run, we simulated 500 times the points and bounds on the interval $[0, 1]$, which was divided uniformly into 200 subintervals. We then determined the average number of times a path crossed the zero level. The empirical distribution of these averages is shown in Fig. 7, (a). The total average for all runs was 0.527874 and the empirical 0.025, 0.5, and 0.975 quantiles were 0.482, 0.528, and 0.568, respectively.

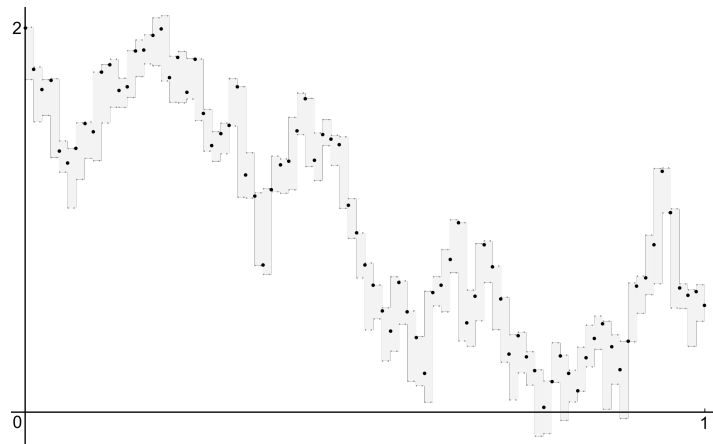


Fig. 6. Simulated path points of the Brownian motion with drift $\alpha = -1$ and volatility $\sigma = \sqrt{5}$

For a series of 50000 simulations, we recorded the value of the left boundary of the subinterval on which the crossing happened. Fig. 7, (b) shows the corresponding histogram and the conditional pdf of the first passage time through level 0, given that the crossing occurred within $[0, 1]$. Given the specified parameter values, and despite the potential for bias, the estimated outcomes appear to be reasonable. For a method of direct and exact simulation of the first-passage time, the reader can refer to (Herrmann, & Zucca, 2019). This method avoids simulating the full path, introduces no approximation error (only statistical error), and does not restrict the first passage time to be within a fixed time interval.

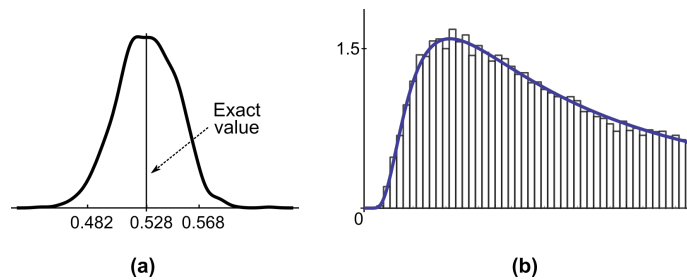


Fig. 7. (a) Kernel-smoothed empirical distribution for the average number of times trajectories cross zero level. (b) Histogram of an approximation of the first passage time and its conditional pdf (solid line)

4. Midpoint simulation

To further detail the trajectory behavior in the interval, we can adapt the idea of the ε -strong algorithm from (Beskos, Peluchetti, & Roberts, 2012). After defining the boundaries, the next step is to determine the position at the middle of the interval. The modeling routine can be justified using the following consequence of (Beskos, Peluchetti, & Roberts, 2012, Proposition 5.1).

Lemma 2. The conditional pdf $f_{mid}(z)$ of $B_{x,l,y}(\frac{l}{2})$ given $B_{x,l,y}^- = u$ and $B_{x,l,y}^+ < v$ is proportional to $p^\pm(z) \phi(z)$, where

$$p^\pm(z) = p^\pm(l, x, y, z, u, v) = -\frac{\partial}{\partial u} \left(\Pr \left\{ B_{x, \frac{l}{2}, z}^- > u, B_{x, \frac{l}{2}, z}^+ < v \right\} \Pr \left\{ B_{z, \frac{l}{2}, y}^- > u, B_{z, \frac{l}{2}, y}^+ < v \right\} \right),$$

$$\phi(z) = e^{-\frac{2}{l} \left(z - \frac{x+y}{2} \right)^2}.$$

For sampling from $f_{mid}(z)$ we can use the accept-reject algorithm with squeezing. First, note that

$$\max_{z \geq u} f_{B_{x,l,x}}^-(u) = \max_{z \geq u} f_{B_{x,l,z}}^-(u) = \begin{cases} \frac{e^{\frac{2(x-u)^2 - l}{l}}}{x-u}, & x-u \leq \sqrt{\frac{l}{2}}, \\ \frac{2(x-u)}{l}, & \text{otherwise.} \end{cases}$$

Second, the monotonicity of probabilities gives $0 \leq -\frac{\partial}{\partial u} \Pr \left\{ B_{x, \frac{l}{2}, z}^- > u, B_{x, \frac{l}{2}, z}^+ < v \right\} \leq f_{B_{x, \frac{l}{2}, z}}^-(u)$. Hence,

$$\begin{aligned} 0 \leq p^\pm(z) &= \left(-\frac{\partial}{\partial u} \Pr \left\{ B_{x, \frac{l}{2}, z}^- > u, B_{x, \frac{l}{2}, z}^+ < v \right\} \right) \Pr \left\{ B_{z, \frac{l}{2}, y}^- > u, B_{z, \frac{l}{2}, y}^+ < v \right\} + \\ &+ \left(-\frac{\partial}{\partial u} \Pr \left\{ B_{z, \frac{l}{2}, y}^- > u, B_{z, \frac{l}{2}, y}^+ < v \right\} \right) \Pr \left\{ B_{x, \frac{l}{2}, z}^- > u, B_{x, \frac{l}{2}, z}^+ < v \right\} \leq \\ &\leq f_{B_{x, \frac{l}{2}, z}}^-(u) + f_{B_{z, \frac{l}{2}, y}}^-(u) \leq \max_{z \geq u} f_{B_{x, \frac{l}{2}, z}}^-(u) + \max_{z \geq u} f_{B_{z, \frac{l}{2}, y}}^-(u) =: C_{x,l,y,u}. \end{aligned} \quad (6)$$

The algorithm requires $0 \leq p^\pm(z) \leq 1$, which can be satisfied using $\frac{1}{C_{x,l,y,u}} p^\pm$ instead of $p^\pm$. Third, the function $\phi(z)$ up to a constant is the density of normal distribution $N\left(\frac{x+y}{2}, \frac{l}{4}\right)$. Finally, without explicitly specifying the dependency on the corresponding arguments, the alternating sequence for $\Pr\{B^- > u, B^+ < v\}$ is $\{h_n\}_{n=1}^\infty$ (for details see (Beskos, Peluchetti, & Roberts, 2012, Section 4)) and for $\left(-\frac{\partial}{\partial u} \Pr\{B^- > u, B^+ < v\}\right)$ is $\{g_n\}_{n=1}^\infty$ where

$$h_{2n-1} = 1 - \sum_{i=1}^n H_i, \quad h_{2n} = \max\left\{1 - \sum_{i=1}^{n-1} H_i - (e^{-q_1^+} + e^{-q_2^+})|_{i=n}, 0\right\},$$

$$g_{2n-1} = \begin{cases} \sum_{i=1}^n G_i + (e^{-q_1^+} p_1^+ + e^{-q_2^+} p_2^+)|_{i=n+1}, & \sum_{i=1}^n G_i \geq 0, \\ f_{B^-}(u), & \text{otherwise,} \end{cases} \quad g_{2n} = \sum_{i=1}^n G_i \mathbb{1}_{\{\sum_{i=1}^n G_i \geq 0\}}.$$

The expression for the lower bound g_{2n} follows directly from Theorem 1. For the upper bound g_{2n-1} we can use arguments similar to those in the proof of the theorem. In case $\sum_{i=1}^n G_i < 0$, instead of the upper bound $f_{B^-}(u)$, a more accurate estimate would be $\left(\sum_{i=1}^n G_i + (e^{-q_1^+} + e^{-q_2^+})|_{i=n+1}\right)$ at the root of the equation $\sum_{i=1}^n G_i = 0$, but finding the corresponding solution at each iteration step can be computationally expensive. Thus, the alternating sequence for $p^\pm$ is defined by $\frac{g_n^{(1)} h_n^{(2)} + g_n^{(2)} h_n^{(1)}}{C_{x,l,y,u}}$ (see Fig. 8), where the number in the superscript indicates the set of terms arguments in (6), $(x, \frac{l}{2}, z, u, v)$ or $(z, \frac{l}{2}, y, u, v)$, respectively.

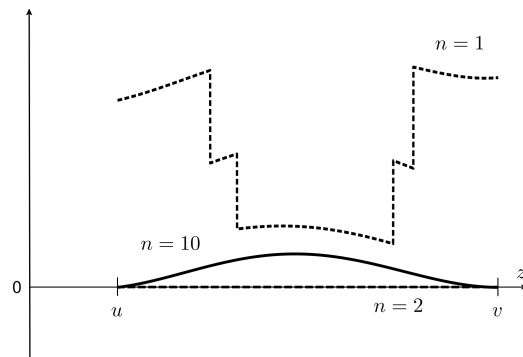


Fig. 8. 1st, 2nd, and 10th elements of the alternating sequence for $p^\pm$ with $l = 1, x = 2.1, y = 2.2, u = 1.8, v = 2.5$

For illustrative purposes, we have included in Fig. 9 the histogram of a sample of 5000 points from $f_{mid}(z)$ with $l = 1, x = 2.1, y = 2.2, u = 1.8$, and $v = 2.5$. The solid curve corresponds to the graph of $\frac{g_n^{(1)} h_n^{(2)} + g_n^{(2)} h_n^{(1)}}{C_{x,l,y,u}}$ with $n = 11$ (a further increase in n does not result in a noticeable change in the graph).

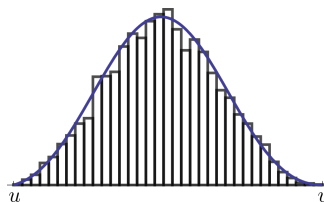


Fig. 9. Histogram of simulated midpoints with $l = 1, x = 2.1, y = 2.2, u = 1.8, v = 2.5$

Remark 3. After bipartition, we take into account that the minimum is reached at least in one of the subintervals. To determine on which one, we simulate the value of the indicator $\mathbb{1}\{\eta < r\}$, where $\eta \sim \text{Uniform}[0, 1]$,

$$r = \frac{\left(-\frac{\partial}{\partial u} \Pr\left\{B_{x, \frac{l}{2}, z}^- > u, B_{x, \frac{l}{2}, z}^+ < v\right\}\right) \Pr\left\{B_{z, \frac{l}{2}, y}^- \geq u, B_{z, \frac{l}{2}, y}^+ < v\right\}}{p^\pm(z)},$$

using the technique from (Beskos, Peluchetti, & Roberts, 2012, Section 4.2). That is, we increase n until $\frac{g_{2n-1}^{(1)} h_{2n-1}^{(2)}}{g_{2n}^{(1)} h_{2n}^{(2)} + g_{2n-1}^{(2)} h_{2n-1}^{(1)}} < \eta$ (and assume that the minimum was reached in the first half of the interval) or $\frac{g_{2n}^{(1)} h_{2n}^{(2)}}{g_{2n-1}^{(1)} h_{2n-1}^{(2)} + g_{2n}^{(2)} h_{2n}^{(1)}} > \eta$ (respectively, in the second). If the denominator is equal to 0, the fraction is defined as 1. For the interval with the given minimum, we can continue to define the value at the midpoint using Lemma 2. For another interval, we can employ the refinement with consequent bisection as described in (Beskos, Peluchetti, & Roberts, 2012, Section 5).

5. Algorithm description

The following basic modeling steps can be obtained by collecting the considered facts together. Let d be the maximum length of subintervals. The upper limits of some loop indices have not been explicitly specified. They can be defined in

accordance with a particular condition. For example, the loop by k can be executed until T_k exceeds the predetermined value. We may continue to execute the loop by m until the accuracy constraints are met.

1. Set $T_0 = 0$, $X_{T_0} = X_0$. Generate J_{T_0} from the initial distribution.

2. Loop by $k \geq 1$.

(a) Draw J_{T_k} from the discrete distribution $\left\{ \frac{Q_{J_{T_{k-1}}j}}{-Q_{J_{T_{k-1}}J_{T_{k-1}}}}, j \neq J_{T_{k-1}} \right\}$.

(b) Generate the inter-jump time $\Delta T_k \sim \text{Exponential}(-Q_{J_{T_{k-1}}J_{T_{k-1}}})$. Set $T_k = T_{k-1} + \Delta T_k$.

(c) Put $t_{k-1}^0 = T_{k-1}$, $n_k = \left\lceil \frac{\Delta T_k}{d} \right\rceil + 1$, $l_k = \frac{\Delta T_k}{n_k}$, where $[\cdot]$ is the integer part.

(d) Loop by $i = \overline{1, n_k}$.

i. Set $t_{k-1}^i = t_{k-1}^{i-1} + l_k$.

ii. Generate

$$X_{t_{k-1}^i} = X_{t_{k-1}^{i-1}} + a_{J_{T_{k-1}}} l_k + \sigma_{J_{T_{k-1}}} \sqrt{l_k} \zeta, \zeta \sim N(0, 1).$$

iii. Generate the minimum $B_{k,i}^-$ and the upper bound of maximum $\check{B}_{k,i}^+$ for Brownian bridge $B_{X_{t_{k-1}^{i-1}}, \sigma_{J_{T_{k-1}}}^2 l_k, X_{t_{k-1}^i}}$ (see Section 3).

iv. Set $s_L = t_{k-1}^{i-1}$, $s_R = t_{k-1}^i$.

v. Loop by $m \geq 2$.

A. Put $s_m = \frac{s_L + s_R}{2}$. Given $B_{k,i}^-$ and $\check{B}_{k,i}^+$ generate midpoint value X_{s_m} using Lemma 2.

B. Choose between intervals $[s_L, s_m]$ and $[s_m, s_R]$ with minimum using Remark 3. Set $s_R = s_m$ in the former case and $s_L = s_m$ in the latter.

C. For the chosen interval, we continue the loop, and for the other interval, we can apply ε -strong algorithm from (Beskos, Peluchetti, & Roberts, 2012).

Remark 4. According to the proposed algorithm, the lower bound for the maximum and the upper bound for the minimum required in the ε -strong algorithm need to be refined only for one of the intervals obtained after midpoint simulation.

Example. Let's choose the following parameters for Z . Assume that for J the state space is $E = \{1, 2\}$, the intensity matrix is $Q = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ 1 & -1 \end{pmatrix}$, and the initial distribution is $\pi = (\frac{2}{3}, \frac{1}{3})$. For X the drift parameters are $a_1 = \frac{1}{2}$, $a_2 = -\frac{3}{2}$, and the volatility parameters are $\sigma_1 = \sqrt{5}$, $\sigma_2 = \sqrt{2}$. Fig. 10 shows the results of applying the algorithm without the ε -strong algorithm steps. Specifically, it displays the simulated trajectory points at the moments of state transitions, the minimum and the upper bound of the maximum on the corresponding intervals. It also displays the midpoints. For the purposes of simplicity, we took $d = \infty$, indicating that $[T_{k-1}, T_k]$ were not divided into subintervals.

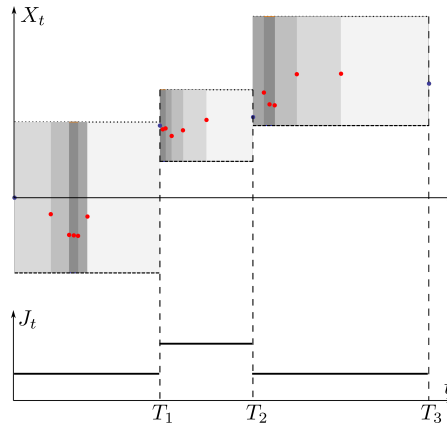


Fig. 10. Regime-switching moments (vertical dashed lines), values of X at those moments (blue points), values of midpoints (red points), the minimum of X (horizontal dotted lines), and the upper bound of maximum (horizontal dashed lines) between regime-switching moments. The gray bars indicate the interval selection (on Step B.). The darker color indicates a later selection

Discussion and conclusions

We propose an algorithm that permits us to generate lower and upper bounds for a fluid flow model of the second order. The algorithm has a direct generalization for models with jumps (under the assumption that the jump part is independent of the diffusion part) and allows for reducing the problem of estimation bias due to discretization. It is based on the ε -strong algorithm incorporating a combination of rejection sampling and quantile transformation.

The algorithm has some pros and cons compared to the ε -strong algorithm. The main advantage is that in the initialization step, we simulate the precise value (up to machine precision) of the minimum and the upper bound of the maximum. By performing additional calculations, it is possible to ascertain a lower bound for the maximum, thereby ensuring the precision of the path upper boundary. The specification of the path behavior at intermediate points with a given value of the minimum (or maximum) necessitates the use of a distinct type of process. For instance, conditional on a Brownian bridge sample path minimum and the time instance when it is achieved, the law of the path parts is that of Bessel bridge (see, e.g., (Pollock, 2013, Section 2.8)). As the value of d decreases, the necessity for intermediate point modeling also decreases; at the same time, the number of calculations increases, which may result in redundancy. The determination of the optimal rule for d depending on the functional under study and the required level of accuracy may be of interest for further research.

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References

- Ahn, S., & Meini, B. (2019). Matrix equations in Markov modulated Brownian motion: theoretical properties and numerical solution. *Stochastic Models*, 36(2), 251–284. <https://doi.org/10.1080/15326349.2019.1704785>
- Almousa, S. A.-D., Horváth, G., & Telek, M. (2020). Transient analysis of piecewise homogeneous Markov fluid models. *Annals of Operations Research*, 310(2), 333–353. <http://dx.doi.org/10.1007/s10479-020-03831-1>
- Beskos, A., Peluchetti, S., & Roberts, G. (2012). ε -strong simulation of the Brownian path. *Bernoulli*, 18(4), 1223–1248. <http://dx.doi.org/10.3150/11-BEJ383>
- Brandimarte, P. (2014). *Handbook in Monte Carlo simulation. Applications in financial engineering, risk management, and economics*. John Wiley & Sons. <http://dx.doi.org/10.1002/9781118593264>
- Breuer, L. (2010a). A quintuple law for Markov additive processes with phase-type jumps. *Journal of Applied Probability*, 47(2), 441–458. <http://dx.doi.org/10.1017/s0021900200006744>
- Breuer, L. (2010b). The total overflow during a busy cycle in a Markov-additive finite buffer system. In Müller-Clostermann, E., & Rathgeb (Eds.), *Measurement, modelling, and evaluation of computing systems and dependability and fault tolerance* (pp. 198–211). Springer. http://dx.doi.org/10.1007/978-3-642-12104-3_16
- Casella, B., & Roberts, G. O. (2008). Exact Monte Carlo simulation of killed diffusions. *Advances in Applied Probability*, 40(1), 273–291. <http://dx.doi.org/10.1239/aap/1208358896>
- Çınlar, E. (1972). Markov additive processes. I. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 24(2), 85–93. <http://dx.doi.org/10.1007/bf00532536>
- D'Auria, B., Ivanovs, J., Kella, O., & Mandjes, M. (2010). First passage of a Markov additive process and generalized Jordan chains. *Journal of Applied Probability*, 47(4), 1048–1057. <http://dx.doi.org/10.1017/s002190020000735x>
- D'Onofrio, G., & Lanteri, A. (2023). Approximating the first passage time density of diffusion processes with state-dependent jumps. *Fractal and Fractional*, 7(1), 30. <http://dx.doi.org/10.3390/fractalfract7010030>
- Doob, J. L. (1949). Heuristic approach to the Kolmogorov-Smirnov theorems. *The Annals of Mathematical Statistics*, 20(3), 393–403. <http://dx.doi.org/10.1214/aoms/117729991>
- Forsythe, G. E., Malcolm, M. A., & Moler, C. B. (1977). *Computer methods for mathematical computations*. Prentice Hall.
- Gribaudo, M., Iacono, M., & Manini, D. (2021). Simulation of n -dimensional second-order fluid models with different absorbing, reflecting and mixed barriers. In *Quantitative evaluation of systems* (pp. 276–292). Springer International Publishing. http://dx.doi.org/10.1007/978-3-030-85172-9_15
- Herrmann, S., & Massin, N. (2023). Exact simulation of the first passage time through a given level of jump diffusions. *Mathematics and Computers in Simulation*, 203, 553–576. <http://dx.doi.org/10.1016/j.matcom.2022.07.007>
- Herrmann, S., & Zucca, C. (2019). Exact simulation of the first-passage time of diffusions. *Journal of Scientific Computing*, 79(3), 1477–1504. <http://dx.doi.org/10.1007/s10915-018-00900-3>
- Horváth, G., & Telek, M. (2017). Matrix-analytic solution of infinite, finite and level-dependent second-order fluid models. *Queueing Systems*, 87(3–4), 325–343. <http://dx.doi.org/10.1007/s11134-017-9544-z>
- Ivanovs, J. (2010). Markov-modulated Brownian motion with two reflecting barriers. *Journal of Applied Probability*, 47(4), 1034–1047. <http://dx.doi.org/10.1017/s0021900200007348>
- Jones, F. (2001). *Lebesgue integration on euclidean space*. Jones and Bartlett Publishers.
- Karandikar, R. L., & Kulkarni, V. G. (1995). Second-order fluid flow models: Reflected Brownian motion in a random environment. *Operations Research*, 43(1), 77–88. <http://dx.doi.org/10.1287/opre.43.1.77>
- Karatzas, I., & Shreve, S. E. (1998). *Brownian motion and stochastic calculus* (2nd ed.). Springer. <http://dx.doi.org/10.1007/978-1-4612-0949-2>
- Latouche, G. (2025). *Markov-modulated Brownian motion. The matrix-analytic perspective*. Springer. <https://link.springer.com/book/9783031985751>
- Latouche, G., & Nguyen, G. T. (2018). Analysis of fluid flow models. *Queueing Models and Service Management*, 1(2), 001-029. https://qmsm.pu.edu.tw/papers/papers/2018_Issue_2/paper_1_Guy_Latouche_p_1_29.pdf
- Metwally, S. A. K., & Atiya, A. (2002). Using Brownian bridge for fast simulation of jump-diffusion processes and barrier options. *The Journal of Derivatives*, 10(1), 43–54. <http://dx.doi.org/10.3905/jod.2002.319189>
- Owen, A. B. (2013). *Monte Carlo theory, methods and examples*. (Accessed: 2025-07-03) <https://artowen.su.domains/mc/>
- Pollock, M. (2013). *Some Monte Carlo methods for jump diffusions* (phdthesis, University of Warwick). <https://wrap.warwick.ac.uk/id/eprint/60602/>
- Pollock, M., Johansen, A. M., & Roberts, G. O. (2016). On the exact and ε -strong simulation of (jump) diffusions. *Bernoulli*, 22(2), 794–856. <http://dx.doi.org/10.3150/14-BEJ676>

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МОДЕЛЮВАННЯ МЕЖ ДЛЯ МОДЕЛІ ПОТОКУ РІДИНИ ДРУГОГО ПОРЯДКУ

Отримано наближену формулу для умовної функції розподілу максимуму броунівського мосту із заданим мінімумом. Формула дає змогу розробити алгоритм моделювання мінімуму та верхньої межі максимуму для траєкторії броунівського мосту. Це, зі свого боку, формує можливість визначити процедуру моделювання для верхньої та нижньої границі для моделі потоку рідини другого порядку. Показано результати роботи алгоритму для певної множини параметрів. Проведено порівняння емпіричних розподілів екстремумів, отриманих за допомогою методу послідовної дискретизації та методу наближених квантилів. Розглянуто можливість моделювання точок посередині інтервалів, а також вибір одного з підінтервалів, на якому досягається мінімум.

Ключові слова: *сильне моделювання, метод відсіювання зі стисканням, квантильне перетворення.*

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