

UDC 517.52

DOI: <https://doi.org/10.17721/1812-5409.2026/1.9>

Mykola MOROZ, PhD, Junior Researcher

ORCID ID: 0000-0001-6658-4924

e-mail: moroznik22@gmail.com

Institute of Mathematics of NAS of Ukraine, Kyiv, Ukraine

THE CONVERGENCE OF SEQUENCES IN TERMS OF POSITIVE AND ALTERNATING PERRON EXPANSIONS

In this paper we study conditions for the convergence of sequences in terms of positive and alternating Perron expansions of real numbers (P -representation and P^- -representation). These conditions make it possible to verify the continuity of functions defined via the P -representation or P^- -representation.

The main results of this work are as follows. First, we systematically consider all possible cases and, by taking into account the geometric structure of Perron expansions, significantly reduce the number of distinct cases that require separate analysis. Second, for each case, we formulate and rigorously prove a convergence condition that is both necessary and sufficient. Third, the results are obtained not for specific expansions (such as the Engel or Pierce expansions), but for the Perron expansions, which include many classical representations of real numbers as special cases.

These results provide a foundation for further analysis of fractal functions defined through Perron expansions or their special cases (the Lüroth, Engel, Pierce, and Sylvester expansions).

Keywords: convergence sequences, Perron expansions, Lüroth expansions, Engel expansions, Pierce expansions.

AMS 2020 classification: 40A05, 11A67, 11K55.

Introduction

Many expansions of real numbers serve as effective tools for defining and analyzing functions that exhibit locally complex behavior and fractal properties, including nowhere monotonic, nowhere differentiable, and singular functions. For instance, various number expansions can be employed to define singular functions such as the Minkowski question-mark function, the Salem function, and the Cantor function, as well as continuous functions that are nowhere monotonic and nowhere differentiable, such as the Tribin function (Prats'ovytyi, Baranovs'kyi, & Maslova, 2021).

In recent years, the Engel and Pierce expansions have been increasingly utilized for such purposes, enabling the modeling of fractal functions with structures more intricate than self-similar ones (see, e.g., (Albeverio et al., 2013; Baranovs'kyi, & Prats'ovytyi, 2023; Moroz, 2017, 2024)). Verifying the continuity of such functions is often essential, yet it is not always a trivial task. However, a general approach to verifying the continuity of such functions was not proposed in these works. In this paper, we aim to lay the foundations of such a general approach. Continuity is understood here in the Heine sense – namely, in terms of the convergence of sequences. Accordingly, our primary focus is on establishing conditions for the convergence of sequences whose elements are given in terms of such expansions.

We formulate and prove all propositions for the positive and alternating Perron expansions of real numbers (P -representations and P^- -representations) (Moroz, 2024, 2025). These expansions generalize a number of classical representations, including the positive and alternating Lüroth expansions (Zhykharyeva, & Prats'ovytyi, 2012; Kalpazidou, Knopfmacher, & Knopfmacher, 1990), Engel expansions (Erdős, Rényi, & Szusz, 1958), Sylvester expansions (Erdős, Rényi, & Szusz, 1958), DKB-expansions (Galambos, 1998), Pierce expansions (Shallit, 1986), and second Ostrogradsky expansions (Torbin, & Prats'ovytyi, 2010). They are also closely related to Oppenheim expansions (Galambos, 1976; Oppenheim, 1972; Sydoruk, & Torbin, 2017).

1. General information about Perron expansions and main notations

Consider functions $\varphi_n(x_1, \dots, x_n): \mathbb{N}^n \rightarrow \mathbb{N}$ for all $n \in \mathbb{N}$, with $\varphi_0 \equiv \text{const} \in \mathbb{N}$. A fixed sequence of such functions $(\varphi_n)_{n=0}^\infty$ is denoted by P . Each sequence P defines two types of Perron expansions of real numbers: the P -representation (positive Perron expansion) and the P^- -representation (alternating Perron expansion).

The P -representation of a number $x \in (0, 1]$ is defined as its expansion into a positive Perron series:

$$x = \sum_{n=0}^{\infty} \frac{r_0 \cdots r_n}{(p_1 - 1)p_1 \cdots (p_n - 1)p_n p_{n+1}}, \quad (1)$$

where $r_0 = \varphi_0$, $r_n = \varphi_n(p_1, \dots, p_n)$, and $p_n \geq r_{n-1} + 1$ for all $n \in \mathbb{N}$. The number x and its P -representation (1) are denoted briefly by $\Delta_{p_1 p_2 \dots}^P$. For each sequence P of functions φ_n , every $x \in (0, 1]$ has a unique P -representation (Moroz, 2024, Theorem 1). The number $p_n = p_n(x)$ is called the n th P -digit of x .

The P -cylinder of rank k with base $c_1 \dots c_k$ is defined as the set $\Delta_{c_1 \dots c_k}^P$ consisting of all numbers $x \in (0, 1]$ such that $p_1(x) = c_1, \dots, p_k(x) = c_k$. It is known (Moroz, 2024, Lemma 3, Corollary 3) that the P -cylinder $\Delta_{c_1 \dots c_k}^P$ is an interval of the form $(a, b]$. The infimum, supremum, and diameter of the P -cylinder $\Delta_{c_1 \dots c_k}^P$ are given by the following formulas:

$$\inf \Delta_{c_1 \dots c_k}^P = \sum_{n=0}^{k-1} \frac{r_0 \cdots r_n}{(c_1 - 1)c_1 \cdots (c_n - 1)c_n c_{n+1}}, \quad (2)$$

$$\sup \Delta_{c_1 \dots c_k}^P = \inf \Delta_{c_1 \dots c_k}^P + \frac{r_0 \cdots r_{k-1}}{(c_1 - 1)c_1 \cdots (c_k - 1)c_k}, \quad (3)$$

$$|\Delta_{c_1 \dots c_k}^P| = \frac{r_0 \dots r_{k-1}}{(c_1 - 1)c_1 \dots (c_k - 1)c_k}, \quad (4)$$

where $r_0 = \varphi_0$ and $r_n = \varphi_n(c_1, \dots, c_n)$ for all $n = 1, \dots, k-1$.

The geometry of the P -representation was studied in greater detail in our previous work (Moroz, 2024). Figures 1 and 2 illustrate the relative positions of P -cylinders and provide insight into the geometric structure of the P -representation.

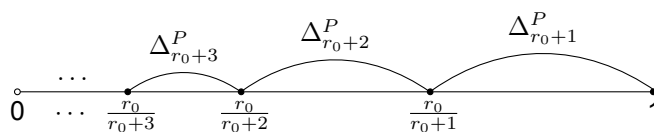


Fig. 1. P -cylinders of rank 1 inside $(0, 1]$, $r_0 = \varphi_0$

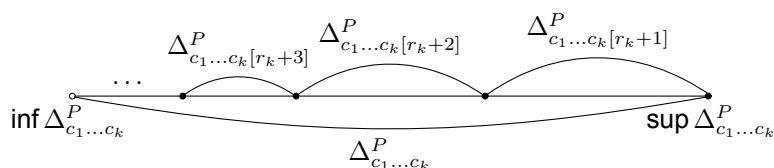


Fig. 2. P -cylinders of rank $k+1$ inside the P -cylinder $\Delta_{c_1 \dots c_k}^P$ of rank k , $r_k = \varphi_k(c_1, \dots, c_k)$

The P^- -representation of a number $x \in (0, 1]$ is defined as its expansion into an alternating Perron series:

$$x = \sum_{n=0}^{\infty} \frac{(-1)^n r_0 \dots r_n}{(q_1 - 1)q_1 \dots (q_n - 1)q_n(q_{n+1} - 1)}, \quad (5)$$

where $r_0 = \varphi_0$, $r_n = \varphi_n(q_1, \dots, q_n)$, and $q_n \geq r_{n-1} + 1$ for all $n \in \mathbb{N}$. The number x and its P^- -representation (5) are denoted briefly by $\Delta_{q_1 q_2 \dots}^{P^-}$. For each sequence P of functions φ_n , there exists a countably infinite set $IS^{P^-} \subset (0, 1]$ whose elements do not have its P^- -representation. However, every $x \in (0, 1) \setminus IS^{P^-}$ has a unique P^- -representation (Moroz, 2025, Theorem 3.12). For $x \in (0, 1) \setminus IS^{P^-}$, the number $q_n = q_n(x)$ is called the n th P^- -digit of x .

The P^- -cylinder of rank k with base $c_1 \dots c_k$ is defined as the set $\Delta_{c_1 \dots c_k}^{P^-}$ of all numbers $x \in (0, 1) \setminus IS^{P^-}$ such that $q_1(x) = c_1, \dots, q_k(x) = c_k$. It is known (Moroz, 2025, Corollary 3.14) that the P^- -cylinder $\Delta_{c_1 \dots c_k}^{P^-}$ is a set of the type $(a, b) \setminus IS^{P^-}$. The infimum, supremum, and diameter of P^- -cylinder $\Delta_{c_1 \dots c_k}^{P^-}$ are given by the following formulas:

- if k is odd, then

$$\sup \Delta_{c_1 \dots c_k}^{P^-} = \sum_{n=0}^{k-1} \frac{(-1)^n r_0 \dots r_n}{(c_1 - 1)c_1 \dots (c_n - 1)c_n(c_{n+1} - 1)}, \quad (6)$$

$$\inf \Delta_{c_1 \dots c_k}^{P^-} = \sup \Delta_{c_1 \dots c_k}^{P^-} - \frac{r_0 \dots r_{k-1}}{(c_1 - 1)c_1 \dots (c_k - 1)c_k}, \quad (7)$$

- if k is even, then

$$\inf \Delta_{c_1 \dots c_k}^{P^-} = \sum_{n=0}^{k-1} \frac{(-1)^n r_0 \dots r_n}{(c_1 - 1)c_1 \dots (c_n - 1)c_n(c_{n+1} - 1)}, \quad (8)$$

$$\sup \Delta_{c_1 \dots c_k}^{P^-} = \inf \Delta_{c_1 \dots c_k}^{P^-} + \frac{r_0 \dots r_{k-1}}{(c_1 - 1)c_1 \dots (c_k - 1)c_k}, \quad (9)$$

- in both cases

$$|\Delta_{c_1 \dots c_k}^{P^-}| = \frac{r_0 \dots r_{k-1}}{(c_1 - 1)c_1 \dots (c_k - 1)c_k}, \quad (10)$$

where $r_0 = \varphi_0$ and $r_n = \varphi_n(c_1, \dots, c_n)$ for all $n = 1, \dots, k-1$. It is important that IS^{P^-} is the set of infima and suprema of all P^- -cylinders.

The geometry of the P^- -representation was studied in greater detail in our previous work (Moroz, 2025). Figures 1, 2, and 3 in (Moroz, 2025) illustrate the relative positions of P^- -cylinders and provide insight into the geometric structure of the P^- -representation. Note that the geometry of the P^- -representation is similar to that of continued fractions.

2. The convergence of sequences in terms of P -representation

We now prove several propositions that establish a connection between the convergence of a numerical sequence and the P -digits of its elements.

Let $x_0 = \Delta_{c_1 c_2 \dots}^P \in (0, 1]$, and let $(x_n)_{n=1}^\infty$ be a sequence of numbers from $(0, 1] \setminus \{x_0\}$. For each $n \in \mathbb{N}$, let k_n denote the smallest index such that $p_{k_n}(x_n) \neq p_{k_n}(x_0)$.

Proposition 1. *If $\lim_{n \rightarrow \infty} k_n = +\infty$, then $\lim_{n \rightarrow \infty} x_n = x_0$.*

Proof. Assume, without loss of generality, that $k_n \geq 2$. Then x_0 and x_n are contained in the same P -cylinder $\Delta_{c_1 \dots c_{k_n-1}}^P$ of rank $k_n - 1$, and we have

$$0 < |x_0 - x_n| < \left| \Delta_{c_1 \dots c_{k_n-1}}^P \right|.$$

Since $k_n \rightarrow +\infty$, according to (Moroz, 2024, Lemma 4) we have $\left| \Delta_{c_1 \dots c_{k_n-1}}^P \right| \rightarrow 0$ as $n \rightarrow \infty$. It follows that $x_n \rightarrow x_0$. $\square$

The condition $\lim_{n \rightarrow \infty} k_n = \infty$ is sufficient for the convergence of the sequence $(x_n)_{n=1}^\infty$ to x_0 . However, it is generally not necessary. There are several fundamental cases, each characterized by its own necessary and sufficient conditions for convergence.

Proposition 2. *If x_0 is not the endpoint of some P -cylinder, then*

$$\lim_{n \rightarrow \infty} x_n = x_0 \iff \lim_{n \rightarrow \infty} k_n = +\infty.$$

Proof. Given the Proposition 1, it suffices to prove that

$$\lim_{n \rightarrow \infty} x_n = x_0 \implies \lim_{n \rightarrow \infty} k_n = +\infty.$$

Suppose to the contrary that $\lim_{n \rightarrow \infty} x_n = x_0$ and $\lim_{n \rightarrow \infty} k_n \neq +\infty$. Then there exists a constant M and an infinite sequence $(n_t)_{t=1}^\infty$ such that $k_{n_t} < M$. Since x_0 is neither the infimum nor the supremum of any P -cylinder, we have

$$\inf \Delta_{c_1 \dots c_M}^P < x_0 = \Delta_{c_1 c_2 \dots}^P < \sup \Delta_{c_1 \dots c_M}^P.$$

However, it follows that $x_{n_t} \notin \Delta_{c_1 \dots c_M}^P$, so

$$|x_0 - x_{n_t}| \geq \min \left\{ x_0 - \inf \Delta_{c_1 \dots c_M}^P, \sup \Delta_{c_1 \dots c_M}^P - x_0 \right\} = \text{const} > 0.$$

Since this inequality holds for infinitely many indices n_t , the sequence $(x_n)_{n=1}^\infty$ cannot converge to x_0 . This contradicts our assumption. Hence, $k_n \rightarrow +\infty$. $\square$

It follows from the geometry of the P -representation that the infimum of any P -cylinder coincides with the supremum of the adjacent left P -cylinder of the same rank. Moreover, the supremum of any P -cylinder, except for $\Delta_{[r_0+1]}^P$, coincides with the infimum of another P -cylinder, though not necessarily of the same rank.

The following proposition provides necessary and sufficient conditions for the sequence $(x_n)_{n=1}^\infty$ to converge to x_0 from the left, where x_0 is the supremum of a P -cylinder.

Proposition 3. *Let $x_0 \in (0, 1]$ be the supremum of a P -cylinder, and let $(x_n)_{n=1}^\infty$ be a sequence such that $0 < x_n < x_0$ for all $n \in \mathbb{N}$. Then*

$$\lim_{n \rightarrow \infty} x_n = x_0 \iff \lim_{n \rightarrow \infty} k_n = +\infty.$$

Proof. The proof is similar to that of Proposition 2. $\square$

The following proposition provides necessary and sufficient conditions for the sequence $(x_n)_{n=1}^\infty$ to converge to x_0 from the right, where x_0 is the infimum of a P -cylinder.

Proposition 4. *Let $x_0 = \inf \Delta_{c_1 \dots c_k}^P$, and let $(x_n)_{n=1}^\infty$ be a sequence such that $x_0 < x_n \leq 1$ for all $n \in \mathbb{N}$. Then $\lim_{n \rightarrow \infty} x_n = x_0$ if and only if there exists an index n_0 such that*

$$\begin{cases} p_i(x_n) = c_i \text{ for all } i \leq k \text{ and } n \geq n_0, \\ \lim_{n \rightarrow \infty} p_{k+1}(x_n) = +\infty. \end{cases} \quad (11)$$

Proof. Sufficiency. It follows from (11) that

$$x_n \in \Delta_{c_1 \dots c_k p_{k+1}(x_n)}^P \subseteq \Delta_{c_1 \dots c_k}^P$$

for all $n \geq n_0$. Thus, $\inf \Delta_{c_1 \dots c_k}^P < x_n \leq \sup \Delta_{c_1 \dots c_k p_{k+1}(x_n)}^P$. Using (Moroz, 2024, Corollary 1), which follows from (2) and (3), we obtain

$$0 < x_n - x_0 \leq \sup \Delta_{c_1 \dots c_k p_{k+1}(x_n)}^P - \inf \Delta_{c_1 \dots c_k}^P = \frac{r_0 r_1 \dots r_k}{(c_1 - 1) c_1 \dots (c_k - 1) c_k (p_{k+1}(x_n) - 1)} \rightarrow 0$$

as $p_{k+1}(x_n) \rightarrow \infty$, where $r_0 = \varphi_0$ and $r_i = \varphi_i(c_1, \dots, c_i)$ for all $i \leq k$. Hence, $x_n \rightarrow x_0$.

Necessity. Assume, for the sake of contradiction, that (11) fails to hold for any n_0 . We distinguish two cases to show that in both $x_n \not\rightarrow x_0$.

Case 1: There exists a sequence $(n_t)_{t=1}^\infty$ and indices $i_t \leq k$ such that $p_{i_t}(x_{n_t}) \neq c_{i_t}$ for all $t \in \mathbb{N}$. In this case, $x_{n_t} \notin \Delta_{c_1 \dots c_k}^P$, so

$$x_0 = \inf \Delta_{c_1 \dots c_k}^P < \sup \Delta_{c_1 \dots c_k}^P < x_{n_t},$$

and therefore

$$x_{n_t} - x_0 > \left| \Delta_{c_1 \dots c_k}^P \right| = \text{const} > 0.$$

Since this inequality holds for infinitely many indices n_t , the sequence $(x_n)_{n=1}^\infty$ does not converge to x_0 .

Case 2: Suppose that $p_i(x_n) = c_i$ for all $i \leq k$ and $n \geq n_0$, but there exist a constant M and a sequence $(n_t)_{t=1}^\infty$ such that $p_{k+1}(x_{n_t}) < M$. Assuming without loss of generality that $n_t \geq n_0$, from (Moroz, 2024, Corollary 1) we have

$$\begin{aligned} x_{n_t} - x_0 &\geq \inf \Delta_{c_1 \dots c_k p_{k+1}(x_{n_t})}^P - \inf \Delta_{c_1 \dots c_k}^P = \frac{r_0 r_1 \dots r_k}{(c_1 - 1) c_1 \dots (c_k - 1) c_k p_{k+1}(x_{n_t})} > \\ &> \frac{r_0 r_1 \dots r_k}{(c_1 - 1) c_1 \dots (c_k - 1) c_k M} = \text{const} > 0, \end{aligned}$$

where $r_0 = \varphi_0$ and $r_i = \varphi_i(c_1, \dots, c_i)$ for all $i \leq k$. Since this inequality holds for infinitely many indices n_t , the sequence $(x_n)_{n=1}^\infty$ does not converge to x_0 .

Thus, condition (11) is both necessary and sufficient for the convergence of $(x_n)_{n=1}^\infty$ to x_0 . $\square$

Since the number 0 does not have a P -representation, we will consider the convergence of the sequence $(x_n)_{n=1}^\infty$ to 0 separately.

Proposition 5. $\lim_{n \rightarrow \infty} x_n = 0 \iff \lim_{n \rightarrow \infty} p_1(x_n) = +\infty$.

Proof. This proposition follows from the inequality $0 < \frac{r_0}{p_1(x_n)} < x_n \leq \frac{r_0}{p_1(x_n) - 1}$, where $r_0 = \varphi_0$. $\square$

3. The convergence of sequences in terms of P^- -representation

Not every number $x \in (0, 1)$ has a P^- -representation. Therefore, we will assume that all elements of the sequence $(x_n)_{n=1}^\infty$ belong to $(0, 1) \setminus IS^{P^-}$. However, the limit of such a sequence can be any number from $[0, 1]$. Consequently, several cases need to be considered.

Let $x_0 = \Delta_{c_1 c_2 \dots}^{P^-} \in (0, 1) \setminus IS^{P^-}$, and let $(x_n)_{n=1}^\infty$ be a sequence of numbers from $(0, 1) \setminus (IS^{P^-} \cup \{x_0\})$. For each $n \in \mathbb{N}$, denote by k_n the smallest index for which $q_{k_n}(x_n) \neq q_{k_n}(x_0)$.

Proposition 6. $\lim_{n \rightarrow \infty} x_n = x_0 \iff \lim_{n \rightarrow \infty} k_n = +\infty$.

Proof. The proof is similar to that of Proposition 1 and Proposition 2. $\square$

Consider the cases in which the sequence $(x_n)_{n=1}^\infty$ converges to a number that does not have a P^- -representation.

Proposition 7. $\lim_{n \rightarrow \infty} x_n = 0 \iff \lim_{n \rightarrow \infty} q_1(x_n) = +\infty$.

Proof. The proof is similar to that of Proposition 5. $\square$

Lemma 1. Every number from IS^{P^-} is the supremum of some P^- -cylinder of odd rank. In particular,

- $\inf \Delta_{c_1 \dots c_k}^{P^-} = \sup \Delta_{c_1 \dots c_{k-1} [c_k+1]}^{P^-}$ if k is odd,
- $\sup \Delta_{c_1 \dots c_k}^{P^-} = \sup \Delta_{c_1 \dots c_k [r_k+1]}^{P^-}$ if k is even, where $r_k = \varphi_k(c_1, \dots, c_k)$,
- $\inf \Delta_{c_1 \dots c_k}^{P^-} = \inf \Delta_{c_1 \dots c_{k-1}}^{P^-} = \sup \Delta_{c_1 \dots [c_{k-1}+1]}^{P^-}$ if k is even and $c_k = \varphi_{k-1}(c_1, \dots, c_{k-1}) + 1$,
- $\inf \Delta_{c_1 \dots c_k}^{P^-} = \sup \Delta_{c_1 \dots c_{k-1} [c_k-1]}^{P^-} = \sup \Delta_{c_1 \dots c_{k-1} [r_k-1] [r_k+1]}^{P^-}$ if k is even and $c_k > \varphi_{k-1}(c_1, \dots, c_{k-1}) + 1$, where $r_k = \varphi_k(c_1, \dots, c_{k-1}, c_k - 1)$.

This lemma follows directly from the geometry of the P^- -representation and from formulas (6), (7), (8), and (9).

Proposition 8. Let $x_0 = \sup \Delta_{c_1 \dots c_k}^{P^-}$, where k is odd, and let $(x_n)_{n=1}^\infty$ be a sequence such that $0 < x_n < x_0$ for all $n \in \mathbb{N}$. Then $\lim_{n \rightarrow \infty} x_n = x_0$ if and only if there exists an index n_0 such that

$$\begin{cases} q_i(x_n) = c_i \text{ for all } i \leq k \text{ and } n \geq n_0, \\ \lim_{n \rightarrow \infty} q_{k+1}(x_n) = +\infty. \end{cases} \quad (12)$$

Proof. Sufficiency. It follows from (12) that

$$x_n \in \Delta_{c_1 \dots c_k q_{k+1}(x_n)}^{P^-} \subseteq \Delta_{c_1 \dots c_k}^{P^-}$$

for all $n \geq n_0$. Then $\inf \Delta_{c_1 \dots c_k q_{k+1}(x_n)}^{P^-} < x_n < \sup \Delta_{c_1 \dots c_k}^{P^-}$. Using (Moroz, 2025, Corollary 3.8), which follows from formulas (6), (7), (8), and (9), we obtain

$$0 < x_0 - x_n < \sup \Delta_{c_1 \dots c_k}^{P^-} - \inf \Delta_{c_1 \dots c_k q_{k+1}(x_n)}^{P^-} = \frac{r_0 r_1 \dots r_k}{(c_1 - 1) c_1 \dots (c_k - 1) c_k (q_{k+1}(x_n) - 1)} \rightarrow 0$$

as $q_{k+1}(x_n) \rightarrow \infty$, where $r_0 = \varphi_0$ and $r_i = \varphi_i(c_1, \dots, c_i)$ for all $i \leq k$. Hence, $x_n \rightarrow x_0$.

Necessity. Assume, for the sake contradiction, that (12) fails to hold for any n_0 . We consider two possible cases and show that in each case $x_n \not\rightarrow x_0$.

Case 1: There exists a sequence $(n_t)_{t=1}^\infty$ and indices $i_t \leq k$ such that $q_{i_t}(x_{n_t}) \neq c_{i_t}$ for all $t \in \mathbb{N}$. In this case, $x_{n_t} \notin \Delta_{c_1 \dots c_k}^{P^-}$, so

$$x_{n_t} < \inf \Delta_{c_1 \dots c_k}^{P^-} < \sup \Delta_{c_1 \dots c_k}^{P^-} = x_0,$$

and therefore

$$x_0 - x_{n_t} > \left| \Delta_{c_1 \dots c_k}^{P^-} \right| = \text{const} > 0.$$

Since this inequality holds for infinitely many indices n_t , the sequence $(x_n)_{n=1}^\infty$ does not converge to x_0 .

Case 2: Suppose that $q_i(x_n) = c_i$ for all $i \leq k$ and $n \geq n_0$, but there exist a constant M and a sequence $(n_t)_{t=1}^\infty$ such that $q_{k+1}(x_{n_t}) < M$. Assuming without loss of generality that $n_t \geq n_0$, from (Moroz, 2025, Corollary 3.8) we have

$$\begin{aligned} x_0 - x_{n_t} &> \sup \Delta_{c_1 \dots c_k}^{P^-} - \sup \Delta_{c_1 \dots c_k q_{k+1}(x_{n_t})}^{P^-} = \frac{r_0 \dots r_k}{(c_1 - 1) c_1 \dots (c_k - 1) c_k q_{k+1}(x_{n_t})} > \\ &> \frac{r_0 \dots r_k}{(c_1 - 1) c_1 \dots (c_k - 1) c_k M} = \text{const} > 0, \end{aligned}$$

where $r_0 = \varphi_0$ and $r_i = \varphi_i(c_1, \dots, c_i)$ for all $i \leq k$. Since this inequality holds for infinitely many indices n_t , the sequence $(x_n)_{n=1}^\infty$ does not converge to x_0 .

Thus, condition (12) is both necessary and sufficient for the convergence of $(x_n)_{n=1}^\infty$ to x_0 . $\square$

Lemma 2. Every number from IS^{P^-} , except for 1, is the infimum of some P^- -cylinder of even rank. In particular,

- $\sup \Delta_{c_1 \dots c_k}^{P^-} = \inf \Delta_{c_1 \dots c_{k-1} [c_k + 1]}^{P^-}$ if k is even,
- $\inf \Delta_{c_1 \dots c_k}^{P^-} = \inf \Delta_{c_1 \dots c_k [r_k + 1]}^{P^-}$ if k is odd, where $r_k = \varphi_k(c_1, \dots, c_k)$,
- $\sup \Delta_{c_1 \dots c_k}^{P^-} = \sup \Delta_{c_1 \dots c_{k-1}}^{P^-} = \inf \Delta_{c_1 \dots [c_{k-1} + 1]}^{P^-}$ if k is odd and $c_k = \varphi_{k-1}(c_1, \dots, c_{k-1}) + 1$,
- $\sup \Delta_{c_1 \dots c_k}^{P^-} = \inf \Delta_{c_1 \dots c_{k-1} [c_k - 1]}^{P^-} = \inf \Delta_{c_1 \dots c_{k-1} [c_k - 1] [r_k + 1]}^{P^-}$ if k is odd and $c_k > \varphi_{k-1}(c_1, \dots, c_{k-1}) + 1$, where $r_k = \varphi_k(c_1, \dots, c_{k-1}, c_k - 1)$.

This lemma follows directly from the geometry of the P^- -representation and from formulas (6), (7), (8), and (9).

Proposition 9. Let $x_0 = \inf \Delta_{c_1 \dots c_k}^{P^-}$, where k is even, and let $(x_n)_{n=1}^\infty$ be a sequence such that $x_0 < x_n < 1$ for all $n \in \mathbb{N}$. Then $\lim_{n \rightarrow \infty} x_n = x_0$ if and only if there exists an index n_0 such that

$$\begin{cases} q_i(x_n) = c_i \text{ for all } i \leq k \text{ and } n \geq n_0, \\ \lim_{n \rightarrow \infty} q_{k+1}(x_n) = +\infty. \end{cases} \quad (13)$$

Proof. The proof is similar to that of Proposition 8. $\square$

Discussion and conclusions

The above propositions provide necessary and sufficient conditions for verifying the convergence of any sequence defined in terms of positive or alternating Perron expansions. If a sequence contains infinitely many elements on both sides of the limit point, which is the endpoint of some P -cylinder or is an element of IS^{P^-} , it suffices to consider two corresponding subsequences and apply the relevant proposition to each of them. Moreover, to establish the continuity of a function at a given point, the propositions concerning one-sided limits are fully sufficient.

Sources of funding. This work was supported by a grant from the Simons Foundation (1290607, M.M.).

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Отримано редакцією журналу / Received: 31.07.25

Прорецензовано / Revised: 30.09.25

Схвалено до друку / Accepted: 16.04.26

Опубліковано / Published: 05.06.26

Микола МОРОЗ, д-р філософії, мол. наук. співроб.

ORCID ID: 0000-0001-6658-4924

e-mail: mykola.moroz@imath.kiev.ua

Інститут математики НАН України, Київ, Україна

ЗБІЖНІСТЬ ЧИСЛОВИХ ПОСЛІДОВНОСТЕЙ У ТЕРМІНАХ ДОДАТНИХ І ЗНАКОЗМІННИХ РОЗКЛАДІВ ПЕРРОНА

Розглянуто умови збіжності числових послідовностей, елементи яких подано у вигляді їхніх додатних і знакозмінних розкладів Перрона (P -представлення та P^- -представлення дійсних чисел відповідно). Ці умови дають можливість дослідити на неперервність функції, що визначені за допомогою P -представлення чи P^- -представлення чисел.

Основні результати дослідження полягають у такому. По-перше, ми систематично розглядаємо всі можливі випадки, та, враховуючи геометричні властивості розкладів Перрона, значно зменшуємо кількість випадків, що потребують окремого розгляду. По-друге, для кожного випадку ми знаходимо та доводимо умову, що є одночасно необхідною та достатньою. По-третє, наші результати отримані не для конкретних розкладів (таких як розклади Енгеля чи Пірса), а для розкладів Перрона загального виду, що охоплюють багато класичних розкладів дійсних чисел (як спеціальні випадки).

Зазначені результати закладають основу для аналізу фрактальних функцій, що визначені в термінах розкладів Перрона чи їхніх часткових випадків (розклади Люрота, Енгеля, Пірса, Сильвестера тощо).

Ключові слова: збіжні послідовності, розклади Перрона, розклади Люрота, розклади Енгеля, розклади Пірса.

Автор заявляє про відсутність конфлікту інтересів. Спонсори не брали участі в розробленні дослідження; у зборі, аналізі чи інтерпретації даних; у написанні рукопису; в рішенні про публікацію результатів.

The author declares no conflicts of interest. The funders had no role in the design of the study; in the collection, analyses or interpretation of data; in the writing of the manuscript; in the decision to publish the results.