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SINGULARITY OF THE POTENTIAL FIELD AT THE VERTEX OF A SYMMETRIC TIP WITH FOUR IDENTICAL PERPENDICULAR WINGS OF WEDGE-SHAPED FORM

The study addresses a spatial problem of the potential theory outside a symmetric tip with four identical perpendicular wings of wedge-shaped form. The main objective is to determine the singularity of the solution at the tip's vertex.

By utilizing the symmetric shape of the tip, the problem is reduced to finding a harmonic function in a quarter-space that satisfies mixed boundary conditions on its sides.

The homogeneous solutions of the problem are expressed in spherical coordinates as a Mehler – Fock-type integral involving Legendre functions. Using the method of integral transforms, the problem is reduced to a Wiener – Hopf-type functional equation, which is transformed into an infinite system of linear algebraic equations by the Wiener – Hopf method. The solutions of this system allow for the derivation of analytical representations for the homogeneous solutions of the original boundary-value problem. The constructed homogeneous solutions enable the determination of the singularity of the potential field at the tip's vertex.

As a result of solving the given problem, a relationship between the singularity exponent and the deviation angle of the tip's wing edges has been established.

The constructed homogeneous solutions enable the determination of the nature of the potential field singularity at the tip's vertex and establish a quantitative relationship between the singularity exponent and the deviation angle of the tip's wing edges. The obtained results provide an assessment of the qualitative behavior of the potential field at the tip's vertex, as well as determine the quantitative characteristics describing this behavior. It makes it possible to find correct solutions of boundary-value problems in complex domains with corner points with the same geometry, especially when numerical methods are used.

Key words: potential theory, mixed boundary-value problem, solution singularity, integral transformation method, Wiener – Hopf technique.

AMS 2020 classification: 31B20, 31B25.

Introduction

Studying physical fields, defining their qualitative and quantitative characteristics, and describing technical and technological processes often involve the analysis of boundary-value problems of mathematical physics in domains with complex shapes. When a field or process is studied in a domain or body with irregular inclusions, or when its boundary contains edges, vertices, or other singular points, the field or process has some distributional singularities in the regions surrounding them, which consequently affect its qualitative and quantitative characteristics. Therefore, identifying the peculiarities of field behavior is an important task from both qualitative and quantitative perspectives with respect to the field or process under study. Furthermore, the identification of these peculiarities makes it possible to improve existing numerical algorithms for solving problems of mathematical physics and to develop new, more advanced ones.

One of the approaches to constructing solutions for problems in domains with internal inclusions and boundary irregularities is based on the use of integral transforms and Wiener–Hopf methods. This approach is quite effective for solving problems in domains of complex geometry. It became especially popular in two-dimensional problems involving the propagation of waves of different types. In particular, the work (Nethercote, Kisil, & Assier, 2022) considers the problem of acoustic wave diffraction on an array of point scatterers. The works (Daniele, & Lombardi, 2021) and (Kuryliak, & Lysechko, 2022) are devoted to the construction of solutions to problems involving the propagation of electromagnetic waves in two-dimensional wedge-shaped domains with boundary conditions of various types. It is also worth paying attention to the works (Nagasaka, Kishiguchi, & Kobayashi, 2022) and (He, & Kobayashi, 2023), in which the authors, using the Wiener – Hopf method, solve two-dimensional problems of electromagnetic wave diffraction on a semi-axis with impedance-type boundary conditions and in a waveguide with a piecewise homogeneous filling. Besides wave problems, the Wiener – Hopf method also makes it possible to solve two-dimensional problems in potential theory and elasticity theory for domains with corner points on the boundary and for domains containing inclusions of irregular geometry (Daniele, & Lombardi, 2022; Loveikin, 2022; 2023). In problems with complex geometry, constructing the exact solutions requires solving the vector Wiener – Hopf equations. The author of the works (Idemen, 1979) presents a method for obtaining the exact solution of such vector equations in some cases. In the works (Kurilyak, 1998; Kurilyak, 1999), vector Wiener – Hopf equations make it possible to construct the solutions of diffraction problems for wedge domains with different boundary conditions.

The present work addresses a spatial problem concerning the singular behavior of a potential field at the vertex of an inclusion shaped as a compound tip composed of four identical sectoral plates (wings), which share a common edge and are arranged in perpendicular planes (see Fig. 1). Locally, at the tip's vertex, the problem exhibits a purely three-dimensional character. Although the nature of the singularity along the wings can be determined using results from two-dimensional potential theory, it is not feasible to do so at the vertex of the tip. Thus, the three-dimensional problem must be solved. A similar conclusion was drawn by Maurizi M. and Berto F. (2020), who investigated spatial singularities in problems of elasticity theory for domains with corner points.

1. Problem formulation

We consider an inclusion shaped as a tip (or spike), with its wings lying in the Oxz i Oyz planes (see Fig. 1). Their common edge lies along the positive semi-axis Oz_+ , while other edges deviate from vertical axis by an angle α ($0 < \alpha \leq \frac{\pi}{2}$).

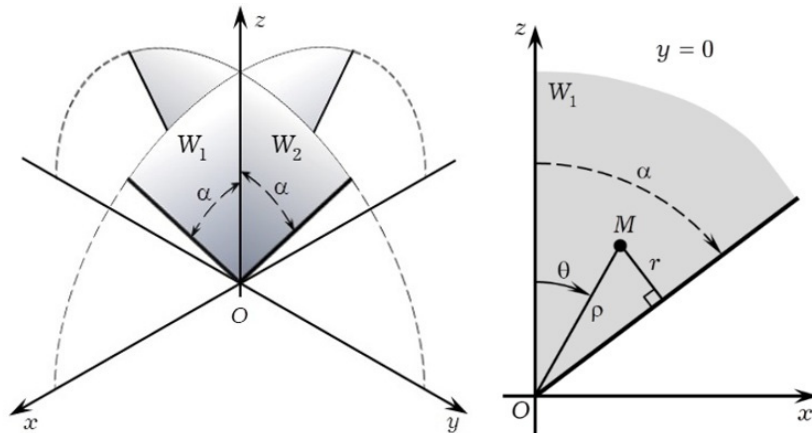


Fig. 1. Form of the tip and wing geometry

To identify the singular behavior of the potential field – electrostatic, for instance – we seek homogeneous solutions of the problem, defined as nontrivial solutions of the corresponding homogeneous boundary-value problem.

$$\Delta u = 0, (x, y, z) \in \mathbb{R}^3 \setminus W, \quad u|_W = 0, \quad (1)$$

where W is the union of all four wings. From a geometrical point of view, the problem is symmetric with respect to the x and y coordinates. Hence, we seek its solution in the coordinate quarter where $x > 0, y > 0$. The main singularity of the behavior of the potential field is determined by the solutions that are even with respect to both the x and y coordinates. For such solutions, the boundary-value problem takes the form:

$$\Delta u = 0, x > 0, y > 0, \quad (2)$$

$$u|_{y=0} = 0, (x, z) \in \overline{W}_1, \quad u|_{x=0} = 0, (y, z) \in \overline{W}_2, \quad (3)$$

$$\frac{\partial u}{\partial y} \Big|_{y=0} = 0, (x, z) \notin \overline{W}_1, \quad \frac{\partial u}{\partial x} \Big|_{x=0} = 0, (y, z) \notin \overline{W}_2, \quad (4)$$

where $W_{1,2}$ are two-dimensional open domains representing the wings lying in the half-planes Ox_+z and Oy_+z respectively (see Fig. 1). Conditions (4) express the evenness of the homogeneous solutions with respect to the y and x coordinates.

Taking into account the geometry of the inclusion, it is more convenient to solve problem (2)–(4) in the spherical coordinate system (ρ, θ, φ) , in which it takes the following form:

$$\Delta u = 0, \rho > 0, 0 < \theta < \pi, 0 < \varphi < \frac{\pi}{2}, \quad (5)$$

$$u|_{\varphi=0} = 0, \rho > 0, 0 < \theta \leq \alpha, \quad \frac{\partial u}{\partial \varphi} \Big|_{\varphi=0} = 0, \rho > 0, \alpha < \theta < \pi. \quad (6)$$

$$u|_{\varphi=\frac{\pi}{2}} = 0, \rho > 0, 0 < \theta \leq \alpha, \quad \frac{\partial u}{\partial \varphi} \Big|_{\varphi=\frac{\pi}{2}} = 0, \rho > 0, \alpha < \theta < \pi. \quad (7)$$

Boundary conditions (6) and (7) are symmetric with respect to the bisectoral half-plane $\varphi = \frac{\pi}{4}$, which divides the domain of the problem (5)–(7) into two symmetric subdomains. As a result, the homogeneous solutions split into two series – even and odd with respect to the half-plane. The main feature of the singular behavior of the potential field at the vertex of the tip is determined by finding even solutions that must satisfy the following condition:

$$\frac{\partial u}{\partial \varphi} \Big|_{\varphi=\frac{\pi}{4}} = 0. \quad (8)$$

The present article is devoted to the construction of such solutions.

2. Construction of homogeneous solutions.

We seek nontrivial solutions of problem (5)–(7), which satisfy condition (8), in the form of a Mehler – Fock-type integral with respect to coordinate θ (Parfenenko, & Ulitko, 1988).

$$u = \rho^{s-\frac{1}{2}} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mu A_s(\mu) \cos\left(\mu\left(\frac{\pi}{4} - \varphi\right)\right) G_s(\mu) P_{s-\frac{1}{2}}^{-\mu}(\cos \theta) d\mu, \rho > 0, 0 < \theta < \pi, 0 \leq \varphi \leq \frac{\pi}{2}, \quad (9)$$

where $A_s(\mu)$ is an unknown coefficient, $G_s(\mu) = \Gamma\left(\frac{1}{2} - s + \mu\right) \Gamma\left(\frac{1}{2} + s + \mu\right)$, $\Gamma(\xi)$ – Euler's gamma function, $P_v^{-\mu}(t)$ is the Legendre function of the first kind (Bateman, & Erdelyi, 1953). The parameter s is assumed to be a purely imaginary number.

The equation (9) can be viewed as Mellin transformation of the desired harmonic function up to a factor $\rho^{s-\frac{1}{2}}$.

For the kernel of the Mehler – Fock-type integral it's possible to obtain the following asymptotic (Bateman, & Erdelyi, 1953)

$$\mu G_s(\mu) P_{s-\frac{1}{2}}^{-\mu}(\cos \theta) \Big|_{\mu \rightarrow \pm i\infty} \sim \Gamma(1 + \mu) \left(\tan \frac{\theta}{2}\right)^\mu, \mu \rightarrow \pm i\infty, s - \text{fixed number};$$

$$\left| \Gamma(1 + \mu) \left(\tan \frac{\theta}{2} \right)^\mu \right|_{\mu=i\tau} = |\Gamma(1 + i\tau)| \sim \sqrt{\tau} e^{-\frac{\pi}{2}\tau}, \quad \tau \rightarrow \pm\infty.$$

With this choice of the function u , condition (8) is automatically satisfied.

Therefore, to determine the unknown coefficient $A_s(\mu)$, we substitute expression (9) into the mixed boundary conditions (6), which are extended over the entire interval $0 < \theta < \pi$ by introducing the unknown functions as follows:

$$\begin{aligned} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mu A_s(\mu) \cos \frac{\pi\mu}{4} G_s(\mu) P_{s-\frac{1}{2}}^{-\mu}(\cos \theta) d\mu &= \begin{cases} 0, & 0 < \theta \leq \alpha, \\ v(\theta), & \alpha < \theta < \pi, \end{cases} \\ \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mu^2 A_s(\mu) \sin \frac{\pi\mu}{4} G_s(\mu) P_{s-\frac{1}{2}}^{-\mu}(\cos \theta) d\mu &= \begin{cases} \sigma(\theta) \sin \theta, & 0 < \theta < \alpha, \\ 0, & \alpha < \theta < \pi. \end{cases} \end{aligned} \quad (10)$$

The unknown function v defines the values of the sought harmonic function u on the extension of the wings, while the unknown function σ represents the normal derivative of u on their surfaces.

We apply direct Mehler – Fock-type transform formulas to equalities (10) (Parfenenko, & Ulitko, 1988).

Using the formula for the Legendre function with a negative argument and expressing these functions in terms of the hypergeometric function (Bateman, & Erdelyi, 1953), we obtain:

$$\begin{aligned} \mu A_s(\mu) \cos \frac{\pi\mu}{4} &= \frac{a^{-\mu} X_s(\mu)}{\Gamma(1 + \mu)}, \\ \mu A_s(\mu) \sin \frac{\pi\mu}{4} &= \frac{1}{\sin(\pi\mu)} \left(-\cos(\pi s) \frac{a^\mu Y_s(\mu)}{\Gamma(1 + \mu)} + \frac{\pi}{G_s(\mu)} \frac{a^{-\mu} Y_s(-\mu)}{\Gamma(1 - \mu)} \right), \quad |Re \mu| < \frac{1}{2}, \end{aligned} \quad (11)$$

where $X_s(\mu), Y_s(\mu)$ are unknown functions of a complex argument, defined in terms of v and σ as follows:

$$\begin{aligned} X_s(\mu) &= \mu \int_0^{\pi-\alpha} \frac{v(\pi-t)}{\sin t} \left(a \cdot \tan \frac{t}{2} \right)^\mu F\left(\frac{1}{2} - s, \frac{1}{2} + s, 1 + \mu, \sin^2 \frac{t}{2}\right) dt, \\ Y_s(\mu) &= \int_0^\alpha \sigma(t) \left(\frac{1}{a} \tan \frac{t}{2} \right)^\mu F\left(\frac{1}{2} - s, \frac{1}{2} + s, 1 + \mu, \sin^2 \frac{t}{2}\right) dt, \end{aligned}$$

$F(\cdot, \cdot, \cdot)$ denotes the hypergeometric function, $a = \tan \frac{\alpha}{2}$ ($0 < a \leq 1$).

Taking into account the root singularity of the harmonic function u along the wing edge and applying Watson's lemma (Olver, 1974), we obtain the following for the unknown functions of a complex argument: $X_s(\mu)$ and $Y_s(\mu)$ are analytic in the half-plane $Re \mu > -\frac{1}{2}$, as functions of the argument μ for each fixed value s , and behave like $\text{Const} \cdot \mu^{-\frac{1}{2}}$ as μ tends to infinity.

Excluding the unknown $A_s(\mu)$ from the equalities (11), we obtain the Wiener–Hopf-type equations (Noble, 1958) with respect to the analytic functions $X_s(\mu), Y_s(\mu)$

$$\frac{G_s(\mu)}{\Gamma(1 + \mu)} \left(X_s(\mu) + \frac{\cos(\pi s) a^{2\mu} Y_s(\mu)}{4 \cos \frac{\pi\mu}{2} \sin^2 \frac{\pi\mu}{4}} \right) = \frac{\pi}{\sin(\pi\mu)} \frac{\cos \frac{\pi\mu}{4}}{\sin \frac{\pi\mu}{4}} \frac{Y_s(-\mu)}{\Gamma(1 - \mu)}, \quad |Re \mu| < \frac{1}{2}. \quad (12)$$

Factorization of the equation coefficient (12) (Bateman, & Erdelyi, 1953)

$$\frac{\pi}{\sin(\pi\mu)} \frac{\cos \frac{\pi\mu}{4}}{\sin \frac{\pi\mu}{4}} = -4\Gamma(\mu)\Gamma(-\mu) \frac{\Gamma\left(1 + \frac{\mu}{4}\right)\Gamma\left(1 - \frac{\mu}{4}\right)}{\Gamma\left(\frac{1}{2} + \frac{\mu}{4}\right)\Gamma\left(\frac{1}{2} - \frac{\mu}{4}\right)}.$$

Applying factorization and separately collecting the functions that are analytic for $Re \mu > -\frac{1}{2}$ and $Re \mu < \frac{1}{2}$, we rewrite equation (12) in the following form:

$$\Phi_s(\mu) + \frac{\cos(\pi s) a^{2\mu}}{4 \cos \frac{\pi\mu}{2} \sin^2 \frac{\pi\mu}{4}} \Psi_s(\mu) = -K_s(-\mu) \Psi_s(-\mu), \quad |Re \mu| < \frac{1}{2}, \quad (13)$$

where, for simplicity of reasoning, we have introduced the following notation for the unknown functions:

$$\begin{pmatrix} \Phi_s(\mu) \\ \Psi_s(\mu) \end{pmatrix} = \frac{\Gamma\left(\frac{1}{2} + \frac{\mu}{4}\right)}{\Gamma\left(1 + \frac{\mu}{4}\right)} \frac{\mu G_s(\mu)}{\Gamma^2(1 + \mu)} \begin{pmatrix} X_s(\mu) \\ Y_s(\mu) \end{pmatrix} \quad (14)$$

and the coefficient:

$$K_s(\mu) = \frac{4\Gamma^2\left(1 + \frac{\mu}{4}\right)\Gamma^2(\mu)}{\Gamma^2\left(\frac{1}{2} + \frac{\mu}{4}\right)G_s(\mu)}.$$

The functions $\Phi_s(\mu), \Psi_s(\mu)$ are analytic in the half-plane $Re \mu > -\frac{1}{2}$, as functions of the variable μ for each fixed value of s . As $\mu \rightarrow \infty$, they behave like $\text{Const} \cdot \mu^{-\frac{1}{2}}$. At the point $\mu = 0$, the functions $\Phi_s(\mu)$ and $\Psi_s(\mu)$ have zeros of at least first order. The coefficient $K_s(\mu)$ has a pole of second order at the point $\mu = 0$, and $K_s(\mu) \rightarrow 1$ as $\mu \rightarrow \infty$.

After factorization, the second term in the left-hand side of equation (13) has a simple pole at $\mu = 0$, which lies in the strip $|Re \mu| < \frac{1}{2}$. This pole will be compensated by the same pole of the function $K_s(-\mu)\Psi_s(-\mu)$ in the right-hand side.

The right-hand side of equation (13) is analytic in the half-plane $Re \mu < \frac{1}{2}$, except for a simple pole at the point $\mu = 0$. The first term on the left-hand side of the equation is analytic for $Re \mu > -\frac{1}{2}$. The second term has simple poles at $\mu = 1 + 2k$, $k = 0, 1, \dots$, second-order poles at $\mu = 4j$, $j = 1, 2, \dots$, and simple pole at $\mu = 0$.

By eliminating these singularities, we can rewrite equation (13) in the following form:

$$\begin{aligned} \Phi_s(\mu) + \frac{\cos(\pi s) a^{2\mu}}{4 \cos \frac{\pi \mu}{2} \sin^2 \frac{\pi \mu}{4}} \Psi_s(\mu) + \frac{\cos(\pi s)}{\pi} \sum_{k=0}^{+\infty} \frac{(-1)^k a^{2(1+2k)}}{\mu - 1 - 2k} x_k - \\ - \frac{4}{\pi^2} \cos(\pi s) \left[\frac{C_0}{\mu} + \sum_{j=1}^{+\infty} a^{8j} \left(\frac{y_j}{(\mu - 4j)^2} + \frac{y_j \ln a^2 + z_j}{\mu - 4j} \right) \right] = -K_s(-\mu) \Psi_s(-\mu) + \frac{\cos(\pi s)}{\pi} \sum_{k=0}^{+\infty} \frac{(-1)^k a^{2(1+2k)}}{\mu - 1 - 2k} x_k - \\ - \frac{4}{\pi^2} \cos(\pi s) \left[\frac{C_0}{\mu} + \sum_{j=1}^{+\infty} a^{8j} \left(\frac{y_j}{(\mu - 4j)^2} + \frac{y_j \ln a^2 + z_j}{\mu - 4j} \right) \right], |Re \mu| < \frac{1}{2}, \end{aligned} \quad (15)$$

where

$$C_0 = \Psi'_s(0), \quad x_k = \Psi_s(1 + 2k), k = 0, 1, \dots, \quad y_j = \Psi_s(4j), \quad z_j = \Psi'_s(4j), j = 1, 2, \dots, \quad (16)$$

and $x_k \sim \frac{\text{Const}}{1+2k}$, $k \rightarrow +\infty$, $y_j \sim \frac{\text{Const}}{j}$, $z_j \sim \frac{\text{Const}}{j^2}$, $j \rightarrow +\infty$.

In the present work, we consider the case where the deviation angle of the tip blades α is bounded by $0 < \alpha \leq \frac{\pi}{2}$. Under these restrictions, the parameter $a = \tan \frac{\alpha}{2}$ belongs to the interval $(0, 1]$, which ensures the convergence of the series in equation (15).

In equality (15), the additional terms were chosen so that the left-hand side is analytic in the half-plane $Re \mu > -\frac{1}{2}$, and the right-hand side is analytic in $Re \mu < \frac{1}{2}$, respectively. As $\mu \rightarrow \infty$, both sides of equation (15) tend to zero as $\text{Const} \cdot \mu^{-1}$. Under these conditions, according to the Wiener–Hopf method (Noble, 1958) both the left-hand and right-hand sides of equation (15) reduce into the zero function. As a result, to determine the unknown analytic functions $\Phi_s(\mu)$, $\Psi_s(\mu)$ we write the following:

$$\begin{aligned} \Phi_s(\mu) + \frac{\cos(\pi s) a^{2\mu}}{4 \cos \frac{\pi \mu}{2} \sin^2 \frac{\pi \mu}{4}} \Psi_s(\mu) + \frac{\cos(\pi s)}{\pi} \sum_{k=0}^{+\infty} \frac{(-1)^k a^{2(1+2k)}}{\mu - 1 - 2k} x_k - \\ - \frac{4}{\pi^2} \cos(\pi s) \left[\frac{C_0}{\mu} + \sum_{j=1}^{+\infty} a^{8j} \left(\frac{y_j}{(\mu - 4j)^2} + \frac{y_j \ln a^2 + z_j}{\mu - 4j} \right) \right] = 0, \quad Re \mu > -\frac{1}{2}; \end{aligned} \quad (17)$$

$$K_s(-\mu) \Psi_s(-\mu) + \frac{\cos(\pi s)}{\pi} \sum_{k=0}^{+\infty} \frac{(-1)^k a^{2(1+2k)}}{\mu + 1 + 2k} x_k + \frac{4}{\pi^2} \cos(\pi s) \left[\frac{C_0}{\mu} + \sum_{j=1}^{+\infty} a^{8j} \left(\frac{y_j}{(\mu + 4j)^2} - \frac{y_j \ln a^2 + z_j}{\mu + 4j} \right) \right] = 0, \quad Re \mu > -\frac{1}{2}. \quad (18)$$

For the complete determination of the analytic functions $\Phi_s(\mu)$ and $\Psi_s(\mu)$ we must find the unknown quantities C_0, x_k , $k = 0, 1, \dots$, y_j, z_j , $j = 1, 2, \dots$, which appear in expressions (17) and (18). To accomplish this, taking into account expression (16), we consecutively substitute $\mu = 1 + 2m$, $m = 0, 1, \dots$ into equality (18). We obtain equations that determine x_k , for $k = 0, 1, \dots$,

$$\begin{aligned} K_s(1 + 2m) x_m + \frac{\cos(\pi s)}{2\pi} \sum_{k=0}^{+\infty} \frac{(-1)^k a^{2(1+2k)}}{1 + m + k} x_k - \\ - \frac{4}{\pi^2} \cos(\pi s) \left[\frac{C_0}{1 + 2m} - \sum_{j=1}^{+\infty} a^{8j} \left(\frac{y_j}{(1 + 2m + 4j)^2} - \frac{y_j \ln a^2 + z_j}{1 + 2m + 4j} \right) \right] = 0, \quad m = 0, 1, 2, \dots \end{aligned} \quad (19)$$

After that, we consecutively substitute $\mu = 4n$, $n = 1, 2, \dots$, into (18). As a result, we obtain equations for determining y_j , $j = 1, 2, \dots$,

$$K_s(4n) y_n + \frac{\cos(\pi s)}{\pi} \sum_{k=0}^{+\infty} \frac{(-1)^k a^{2(1+2k)}}{4n + 1 + 2k} x_k - \frac{\cos(\pi s)}{\pi^2} \left[\frac{C_0}{n} - \sum_{j=1}^{+\infty} a^{8j} \left(\frac{y_j}{4(n + j)^2} - \frac{y_j \ln a^2 + z_j}{n + j} \right) \right] = 0, \quad n = 1, 2, \dots \quad (20)$$

Finally, we differentiate expression (18) and consecutively substitute $\mu = 4n$, $n = 1, 2, \dots$, into the resulting equality. Hence, we obtain equations that define the unknown values z_j for $j = 1, 2, \dots$,

$$\begin{aligned} K_s(4n) \left(z_n + \left\{ \frac{1}{2} \left(\psi(n) - \psi\left(\frac{1}{2} + n\right) \right) + 2\psi(1 + 4n) - \psi\left(\frac{1}{2} - s + 4n\right) - \psi\left(\frac{1}{2} - s + 4n\right) \right\} y_n \right) - \\ - \frac{\cos(\pi s)}{\pi} \sum_{k=0}^{+\infty} \frac{(-1)^k a^{2(1+2k)}}{(4n + 1 + 2k)^2} x_k + \frac{\cos(\pi s)}{4\pi^2} \left[\frac{C_0}{n^2} - \sum_{j=1}^{+\infty} a^{8j} \left(\frac{y_j}{2(n + j)^3} - \frac{y_j \ln a^2 + z_j}{(n + j)^2} \right) \right] = 0, \quad n = 1, 2, \dots, \end{aligned} \quad (21)$$

Here, $\psi(t)$ denotes the logarithmic derivative of the gamma function (Bateman, & Erdelyi, 1953).

Equations (19)–(21) do not close the infinite system with respect to all unknown values, as they don't include an equation for determining coefficient C_0 . To obtain the final missing equation, we express the function $\Psi_s(\mu)$ from equality (18). The resulting expression is then substituted into equation (17), and we take the limit as $\mu \rightarrow 0$. Thus, we obtain the last equation of the system.

$$\left[\cos(\pi s) \left(\ln a^2 + \frac{1}{2} \psi\left(\frac{1}{2}\right) + 2\psi\left(\frac{1}{2} + s\right) - \frac{5}{2} \psi(1) \right) - \pi \sin(\pi s) \right] C_0 - \quad (22)$$

$$-\frac{\pi}{2} \cos(\pi s) \sum_{k=0}^{+\infty} \frac{(-1)^k a^{2(1+2k)}}{1+2k} x_k - \frac{\cos(\pi s)}{4} \sum_{j=1}^{+\infty} a^{8j} \left(\frac{y_j}{2j^2} - 2 \frac{y_j \ln a^2 + z_j}{j} \right) = 0.$$

As a result of the solution, the Wiener – Hopf-type equation (17) has been reduced to an infinite system of linear algebraic equations (19) – (22). In the case $0 < a = \tan \frac{\alpha}{2} < 1$ ($0 < \alpha < \frac{\pi}{2}$), the resulting system is quasi-regular with meromorphic coefficients for all complex values of s . In the case $a = \tan \frac{\alpha}{2} = 1$ ($\alpha = \frac{\pi}{2}$), the reduction to the quasi-regular system (19)–(22) can be achieved by the change of variables: $x_k = \frac{\tilde{x}_k}{\sqrt{1+2k}}$, $k = 0, 1, \dots$, $y_j = \frac{\tilde{y}_j}{\sqrt{j}}$, $z_j = \frac{\tilde{z}_j}{\sqrt{j}}$, $j = 1, 2, \dots$. The regularity of the infinite system (19)–(22) allows us to analytically continue it with respect to the parameter s from the imaginary axis to the entire complex plane. For the analysis of the solutions of this system and their numerical approximation (if needed), the method of reduction to the finite system can be applied.

4. Explicit form of the homogeneous solution and analysis of the field singularity at the tip vertex.

For the final construction of the homogeneous solutions, we write their representation in terms of the solutions of the infinite system of linear algebraic equations (19)–(22). For this purpose, we determine the coefficient $A_s(\mu)$ from the first equality in (11) and substitute it into expression (9). Taking into account equality (14), we obtain:

$$u = \frac{\rho^{s-\frac{1}{2}}}{2\pi i} \int_{-i\infty}^{+i\infty} a^{-\mu} \Phi_s(\mu) \Gamma(\mu) \frac{\Gamma\left(\frac{1}{2} + \frac{\mu}{4}\right) \cos\left(\mu\left(\frac{\pi}{4} - \varphi\right)\right)}{\Gamma\left(1 + \frac{\mu}{4}\right) \cos\frac{\pi\mu}{4}} P_{s-\frac{1}{2}}^{-\mu}(\cos \theta) d\mu, \quad \rho > 0, 0 < \theta < \pi, 0 \leq \varphi \leq \frac{\pi}{2}. \quad (23)$$

In the case $0 < \theta \leq \alpha$, the integral (23) can be evaluated using the Residue theorem. If we close the integration contour in the half-plane $\operatorname{Re} \mu > 0$ and compute sum of residues at all simple poles $\mu = 2 + 4l$, $l = 0, 1, \dots$, of the integrand in this half-plane, then we obtain:

$$u = \frac{4}{\pi} \rho^{s-\frac{1}{2}} \sum_{l=0}^{+\infty} \Phi_s(2+4l) \frac{\Gamma(1+l) \Gamma(2+4l)}{\Gamma\left(\frac{3}{2}+l\right) a^{2+4l}} P_{s-\frac{1}{2}}^{-2-4l}(\cos \theta) \sin((2+4l)\varphi), \quad \rho > 0, 0 < \theta \leq \alpha, 0 \leq \varphi \leq \frac{\pi}{2}. \quad (24)$$

From equalities (17), (18) it follows that the values $\Phi_s(2+4l)$, $l = 0, 1, \dots$, are expressed linearly in terms of the solutions of the infinite system (19)–(22): $C_0, x_k, k = 0, 1, \dots, y_j, z_j, j = 1, 2, \dots$. Thus, the explicit form of the sought homogeneous solutions of the boundary-value problem (5) – (8) is completely defined by the solutions of the system (19)–(22).

A similar representation can also be obtained for the homogeneous solutions in the interval $\alpha < \theta < \pi$.

Up to this point, we have assumed that the parameter s is purely imaginary. Since both the infinite system (19)–(22) and its solutions can be analytically continued from the imaginary axis to the entire complex plane, representation (24) remains valid for the sought homogeneous solutions for all arbitrary complex values of s .

The system (19)–(22) is a linear homogeneous system. Therefore, it has the obvious trivial solution for all complex values of s . However, to determine the homogeneous solutions of the boundary-value problem (5)–(8), this system must have a nontrivial solution, which is only possible for certain values of the parameter s . These specific values of s determine the final form of the homogeneous solutions of problem (5)–(8). Since the infinite system of linear algebraic equations (19)–(22) is quasi-regular for all complex values of s , it can be replaced by a reduced (finite) system for the approximate determination of the possible values of s . The reduced system will have nontrivial solutions if and only if its determinant is equal to zero. The numerical analysis shows that the parameter s can only take real values symmetric with respect to zero.

The singularity of the potential field at the tip vertex at the point O (see Fig. 1) is determined by positive values of s . If we consider only the homogeneous solution corresponding to the smallest positive value $s = s_1$, then, taking into account expression (24), the derivative of the potential on blade W_1 (see Fig. 1) along the normal $\vec{n}$ can be written as follows:

$$\left. \frac{\partial u}{\partial \vec{n}} \right|_{W_1} = \frac{1}{\rho \sin \theta} \left. \frac{\partial u}{\partial \varphi} \right|_{\varphi=0} = \frac{4}{\pi} \rho^{\gamma_1} \sum_{l=0}^{\infty} \frac{\Phi_{s_1}(2+4l)}{\sin \theta} \frac{\Gamma(1+l) \Gamma(3+4l)}{\Gamma\left(\frac{3}{2}+l\right) a^{2+4l}} P_{s_1-\frac{1}{2}}^{-2-4l}(\cos \theta), \quad \rho > 0, 0 < \theta < \alpha, \quad (25)$$

where $\gamma_1 = s_1 - \frac{3}{2}$.

It follows from (17) and (18) that the values $\Phi_{s_1}(2+4l)$, $l = 0, 1, 2, \dots$, satisfy the following asymptotic expressions:

$$\Phi_{s_1}(2+4l) \sim \frac{F_1}{2+4l}, \quad l \rightarrow \infty, \quad F_1 = \text{Const.}$$

Using it together with the properties of Gamma function and Legendre function $P_v^{-\mu}(\cos \theta)$ (Bateman, & Erdelyi, 1953), expression (25) could be rewritten as follows:

$$\left. \frac{\partial u}{\partial \vec{n}} \right|_{W_1} = \rho^{\gamma_1} \left(\frac{Q_1 \sin \theta}{\sqrt{\sin(\alpha - \theta)}} + w_1(\theta) \right), \quad \rho > 0, 0 < \theta < \alpha, \quad (26)$$

where $Q_1 = \text{Const}$, and $w_1(\theta)$ denotes the sum of a uniformly convergent series on $0 \leq \theta \leq \alpha$ and, therefore, is a continuous function.

The following Tab. 1 represents approximate values of s_1 and the singularity exponent depending on the angle deviation α of the blade edges.

Analyzing equality (26) and numerical results presented in the table, we can conclude that as the tip vertex (point O , see Fig. 1) is approached while $\rho \rightarrow 0$, and $\theta \in (0, \alpha)$ is a fixed value, the derivative of the potential along the normal grows like

$$\left. \frac{\partial u}{\partial \vec{n}} \right|_{W_1} \sim K_0(\theta) \rho^{\gamma_1}, \quad \rho \rightarrow 0+.$$

The obtained result shows that the potential field has a power-law singularity at the tip vertex, with the exponent γ_1 depending on the deviation angle α of the wing edges. For sufficiently sharp deviation angles of the edges $0 < \alpha < \alpha_*$

$\approx 0.36894 \pi$ the singularity exponent satisfies the estimate $\gamma_1 < -\frac{1}{2}$. This means that the power-law singularity at the tip vertex is stronger than the classical power-law singularity that occurs along the wing edges. In the case $\alpha = \alpha_*$, the classical square-root singularity occurs at the tip vertex, while for $\alpha_* < \alpha \leq \frac{\pi}{2}$ the singularity at the vertex becomes weaker than that along the wing edges.

Table 1

Approximate values of s_1 – the smallest positive zero of the determinant of the reduced system (19)–(22) and the singularity exponent γ_1 of the solution's normal derivative

α/π	s_1	γ_1	α/π	s_1	γ_1	α/π	s_1	γ_1
0.01	0.61075	– 0.88926	0.2	0.81605	– 0.68395	0.36894	1	– 0.5
0.05	0.671	– 0.82901	0.25	0.8651	– 0.63493	0.4	1.0416	– 0.45836
0.1	0.7224	– 0.77761	0.3	0.91795	– 0.58209	0.45	1.1168	– 0.38332
0.15	0.76925	– 0.73079	0.35	0.97615	– 0.52385	0.5	1.2103	– 0.29568

Based on equality (26), we can analyze the behavior of the square-root singularity intensity coefficient that arise along the wing edges. If we fix the value of ρ and let θ approach $\alpha -$, then from (26) we obtain

$$\left. \frac{\partial u}{\partial n} \right|_{w_1} \sim \frac{K_1(\rho)}{\sqrt{r}}, \theta \rightarrow \alpha -,$$

where $r = \rho \sin(\alpha - \theta)$ is the distance to the edge (see Fig.1), and the following equality holds for the intensity coefficient of the singularity:

$$K_1(\rho) = \rho^{\gamma_1 + \frac{1}{2}} Q_1 \sin \alpha, \quad \rho > 0.$$

From the obtained equality for the coefficient K_1 and numerical results presented in the table, we conclude that for sharp angles $0 < \alpha < \alpha_* \approx 0.36894 \pi$, the intensity coefficient of the singularity on the edge tends to infinity as the tip vertex O is approached along the edge. This is due to the fact that the singularity at the tip vertex is stronger than those along the edges. In the case $\alpha = \alpha_*$, when the classical singularity occurs at point O , the coefficient K_1 remains bounded while approaching this point. On the other hand, if $\alpha_* < \alpha \leq \frac{\pi}{2}$, the intensity coefficient tends to zero as the tip vertex O is approached, since for such values of the angle α , the singularity at the vertex is weaker than that along the wing edges.

Discussion and conclusions

In the present work, an exact solution to the three-dimensional problem of potential theory was obtained for determining the singularity of the potential field at the tip vertex formed by four mutually perpendicular wedge-shaped blades sharing a common edge.

To construct the homogeneous solutions of the problem that make it possible to identify the singularity, spherical coordinates are used together with Mehler–Fock-type integral transforms involving Legendre functions.

The problem was reduced to a Wiener–Hopf-type functional equation, which, using the Wiener–Hopf method, was further reduced to a quasi-completely regular infinite system of linear algebraic equations. The analysis of the obtained infinite system allowed us to determinate the explicit form of the homogeneous solutions of the problem.

The constructed homogeneous solutions make it possible to determine and analyze the singularity of the potential field at the tip vertex. It has been established that the behavior of the potential field near the tip vertex has a complex three-dimensional character, and the singularity of this behavior can only be determined by solving the spatial problem considered in the present work.

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ОСОБЛИВІСТЬ ПОТЕНЦІАЛЬНОГО ПОЛЯ У ВЕРШИНІ СИМЕТРИЧНОГО НАКОНЕЧНИКА ІЗ ЧОТИРМА ОДНАКОВИМИ ПЕРПЕНДИКУЛЯРНИМИ КЛИНОПОДІБНИМИ КРИЛАМИ

Розв'язано просторову задачу теорії потенціалу зовні включення у формі симетричного наконечника із чотирма однаковими перпендикулярними крилами клиноподібної форми. Основна мета – визначення особливості поведінки розв'язку у вершині наконечника.

Використовуючи симетричну форму наконечника, задачу зведено до відшукування гармонічної функції у чверті простору, що має задовольняти мішані крайові умови на його сторонах.

Однорідні нетривіальні розв'язки задачі подано у сферичних координатах інтегралом типу Мелера – Фока по функціях Лежандра. Методом інтегральних перетворень задачу зведено до функціонального рівняння типу Вінера – Гопфа, що методом Вінера – Гопфа зводиться до однорідної нескінченної системи лінійних алгебраїчних рівнянь. Розв'язки отриманої системи дають змогу записати аналітичні зображення для однорідних розв'язків вихідної крайової задачі.

Побудовані однорідні розв'язки допомагають встановити характер особливості потенціального поля у вершині наконечника. Показники особливості визначаються з умов існування нетривіальних розв'язків однорідної нескінченної системи лінійних алгебраїчних рівнянь.

Унаслідок розв'язання поставленої задачі встановлено залежність між показником особливості та кутом відхилення ребер крил наконечника. Отримані результати дають змогу оцінити якісну картину поведінки потенціального поля у вершині наконечника, а також визначити кількісні характеристики, що описують цю поведінку.

Ключові слова: теорія потенціалу, мішана крайова задача, особливість розв'язку, метод інтегральних перетворень, метод Вінера – Гопфа.

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