

UDC 519.2

DOI: <https://doi.org/10.17721/1812-5409.2026/1.3>

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DYNAMICS OF A NON-AUTONOMOUS STOCHASTIC MUTUALISM MODEL WITH JUMPS

In nature, we can find many examples where the interaction of two or more species is to the advantage of all. Mutualism occurs when one species provides some benefit in exchange for some benefit, for example, ants and aphids, in which the ants obtain honeydew food resources excreted by aphids while the aphids obtain increased survival by the non-trophic service of ant defense against natural enemies of the aphids. Mathematical deterministic mutualism models are widely used to study the dynamics of such population systems. In real life, there is a lot of randomness and stochasticity, such as environmental noise, so using stochastic models is more suitable. Suppose we want to take into account abrupt environmental perturbations, such as epidemics, fires, earthquakes, etc., in the considered models. In that case, we must introduce Poisson noises into population models to describe such discontinuous systems. So, we take into account not only "small" jumps, corresponding to the centered Poisson measure, but also the "large" jumps, corresponding to the non-centered Poisson measure. The existence and uniqueness of the global positive solution are proved for the system of stochastic differential equations describing a non-autonomous mutualism model disturbed by white noise, centered, and non-centered Poisson noises. We obtain sufficient conditions of stochastic ultimate boundedness, stochastic permanence, non-persistence in the mean, weak and strong persistence in the mean, and extinction of the solution to the considered system.

Key words: *stochastic mutualism model, global solution, stochastic ultimate boundedness, stochastic permanence, extinction, non-persistence in the mean, weak and strong persistence in the mean.*

AMS 2020 classification: 92D25, 60H10, 60H30.

Introduction

In nature we can find many examples where the interaction of two or more species is to the advantage of all. Mutualism occurs when one species provides some benefit in exchange for some benefit, for example, ants and aphids, in which the ants obtain honeydew food resources excreted by aphids while the aphids obtain increased survival by the non-trophic service of ant defense against natural enemies of the aphids. Mathematical deterministic mutualism models are widely used to study the dynamics of such population systems.

In (May, 1976), the following deterministic two-species mutualism model

$$\begin{aligned} dx(t) &= x(t) \left(r_1 - \frac{b_1 x(t)}{K_1 + y(t)} - \varepsilon_1 x(t) \right) dt, \\ dy(t) &= y(t) \left(r_2 - \frac{b_2 y(t)}{K_2 + x(t)} - \varepsilon_2 y(t) \right) dt \end{aligned} \quad (1)$$

is proposed. Here $x(t), y(t)$ denote the population densities of each species at time t , $r_i, K_i, b_i, \varepsilon_i$ ($i = 1, 2$) are positive constants, r_1, r_2 denote the intrinsic growth rate of species $x(t), y(t)$ respectively, K_1 is the capability of species $x(t)$ being short of $y(t)$, similarly K_2 is the capability of species $y(t)$ being short of $x(t)$.

In the paper (Li et al., 2015) the stochastic version of the model (1) is considered in the following form

$$\begin{aligned} dx(t) &= x(t) \left(r_1 - \frac{b_1 x(t)}{K_1 + y(t)} - \varepsilon_1 x(t) \right) dt + \alpha_1 x(t) dw_1(t), \\ dy(t) &= y(t) \left(r_2 - \frac{b_2 y(t)}{K_2 + x(t)} - \varepsilon_2 y(t) \right) dt + \alpha_2 y(t) dw_2(t), \end{aligned} \quad (2)$$

where $w_1(t), w_2(t)$ are mutually independent Wiener processes. The authors proved the existence and uniqueness of a global positive solution to the system (2). It is proven that the solution to the system (2) is stochastically ultimate bounded, stochastically permanent, uniformly Hölder-continuous. The sufficient conditions for persistence in the mean and extinction of the solution to the considered system are obtained.

Population systems may suffer abrupt environmental perturbations, such as epidemics, fires, earthquakes, etc. So, it is natural to introduce Poisson noises into the population model for describing such discontinuous systems. In the paper (Li, Gao, & Wang, 2016), the authors consider the stochastic mutualism model with white and centered Poisson noises

$$\begin{aligned} dx(t) &= x(t^-) \left[\left(r_1(t) - \frac{b_1(t)x(t)}{K_1(t)+y(t)} - \varepsilon_1(t)x(t) \right) dt + \alpha_1(t)dw_1(t) + \int_{\mathbb{Y}} \gamma_1(t, z) \tilde{\nu}(dt, dz) \right], \\ dy(t) &= y(t^-) \left[\left(r_2(t) - \frac{b_2(t)y(t)}{K_2(t)+x(t)} - \varepsilon_2(t)y(t) \right) dt + \alpha_2(t)dw_2(t) + \int_{\mathbb{Y}} \gamma_2(t, z) \tilde{\nu}(dt, dz) \right]. \end{aligned} \quad (3)$$

Here $x(t^-)$, $y(t^-)$ are the left limit of $x(t)$ and $y(t)$ respectively, $r_i(t)$, $b_i(t)$, $K_i(t)$, $\varepsilon_i(t)$, $\alpha_i(t)$, $i = 1, 2$ are all positive, continuous and bounded functions, $\mathbb{Y}$ is a measurable subset of $(0, +\infty)$, $w_i(t)$, $i = 1, 2$ are independent standard one-dimensional Wiener processes, $\tilde{\nu}(t, A) = \nu(t, A) - t\Pi(A)$ is the centered Poisson measure independent on $w_i(t)$, $i = 1, 2$, $E[\nu(t, A)] = t\Pi(A)$, $\Pi(\mathbb{Y}) < \infty$, $\gamma_i(t, z)$, $i = 1, 2$ are random, measurable, bounded, continuous on t . The global existence and uniqueness of the positive solution to the problem (3) are proved. The sufficient conditions of stochastic ultimate boundedness, stochastic permanence, persistence in the mean, and extinction of the solution to the system (3) are obtained.

In this paper, we consider the stochastic mutualism model with jumps generated by centered and non-centered Poisson measures. So, we take into account not only "small" jumps, corresponding to the centered Poisson measure, but also the "large" jumps, corresponding to the non-centered Poisson measure. This model is driven by the system of stochastic differential equations

$$\begin{aligned} dx_i(t) = & x_i(t) \left[r_i(t) - \frac{b_i(t)x_i(t)}{K_i(t) + x_{3-i}(t)} - c_i(t)x_i(t) \right] dt + \sigma_i(t)x_i(t)dw_i(t) + \\ & + \int_{\mathbb{R}} \gamma_i(t, z)x_i(t^-)\tilde{\nu}_1(dt, dz) + \int_{\mathbb{R}} \delta_i(t, z)x_i(t^-)\nu_2(dt, dz), \quad x_i(0) = x_{i0} > 0, \quad i = 1, 2, \end{aligned} \quad (4)$$

where $w_i(t)$, $i = 1, 2$ are independent standard one-dimensional Wiener processes, $\tilde{\nu}_1(t, A) = \nu_1(t, A) - t\Pi_1(A)$, $\nu_i(t, A)$, $i = 1, 2$ are independent Poisson measures, which are independent on $w_i(t)$, $i = 1, 2$, $E[\nu_i(t, A)] = t\Pi_i(A)$, $i = 1, 2$, $\Pi_i(A)$, $i = 1, 2$ are finite measures on the Borel sets A in $\mathbb{R}$.

We prove that system (4) has a unique, positive, global solution for any positive initial value, and this solution is stochastically ultimately bounded. The sufficient conditions for stochastic permanence, extinction, non-persistence in the mean, weak and strong persistence in the mean of the solution to the system (4) are derived.

It is worth noting that the impact of centered and non-centered Poisson noises to the stochastic predator-prey model that depends on the population density of the predator is studied in the paper (Borysenko, & Borysenko, 2023).

In the following we will use the notations $X(t) = (x_1(t), x_2(t))$, $X_0 = (x_{10}, x_{20})$, $|X(t)| = \sqrt{x_1^2(t) + x_2^2(t)}$, $\mathbb{R}_+^2 = \{X \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$,

$$\beta_i(t) = \frac{\sigma_i^2(t)}{2} + \int_{\mathbb{R}} [\gamma_i(t, z) - \ln(1 + \gamma_i(t, z))] \Pi_1(dz) - \int_{\mathbb{R}} \ln(1 + \delta_i(t, z)) \Pi_2(dz),$$

$i = 1, 2$. For the bounded, continuous functions $f_i(t)$, $t \in [0, +\infty)$, $i = 1, 2$, let us denote

$$f_i \sup = \sup_{t \geq 0} f_i(t), \quad f_i \inf = \inf_{t \geq 0} f_i(t), \quad i = 1, 2, \quad f_{\max} = \max\{f_1 \sup, f_2 \sup\}, \quad f_{\min} = \min\{f_1 \inf, f_2 \inf\}.$$

The rest of this paper is organized as follows. In Section 2, we prove the existence of a unique global positive solution to the system (4) and derive some auxiliary results. In Section 3, we prove the stochastic ultimate boundedness of the solution to the system (4), and obtain sufficient conditions under which the solution is stochastically permanent. The sufficient conditions for extinction, non-persistence in the mean, strong and weak persistence in the mean of the solution are derived.

1. Existence of a global solution and some auxiliary lemmas

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $w_i(t)$, $i = 1, 2$, $t \geq 0$ are independent standard one-dimensional Wiener processes on $(\Omega, \mathcal{F}, P)$, and $\nu_i(t, A)$, $i = 1, 2$ are independent Poisson measures defined on $(\Omega, \mathcal{F}, P)$ independent on $w_i(t)$, $i = 1, 2$. Here $E[\nu_i(t, A)] = t\Pi_i(A)$, $i = 1, 2$, $\tilde{\nu}_i(t, A) = \nu_i(t, A) - t\Pi_i(A)$, $i = 1, 2$, $\Pi_i(\cdot)$, $i = 1, 2$ are finite measures on the Borel sets in $\mathbb{R}$. On the probability space $(\Omega, \mathcal{F}, P)$ we consider an increasing, right continuous family of complete sub- σ -algebras $\{\mathcal{F}_t\}_{t \geq 0}$, where $\mathcal{F}_t = \sigma\{w_i(s), \nu_i(s, A), s \leq t, i = 1, 2\}$.

We need the following assumption.

Assumption 1. It is assumed, that $r_i(t) > 0$, $b_i(t) > 0$, $K_i(t) > 0$, $c_i(t) > 0$, $\sigma_i(t)$, $i = 1, 2$ are bounded, continuous on t functions, $c_{\min} > 0$, $K_{\min} > 0$, $\gamma_i(t, z)$, $\delta_i(t, z)$, $i = 1, 2$ are continuous on t functions and $\ln(1 + \gamma_i(t, z))$, $\ln(1 + \delta_i(t, z))$ are well-defined and bounded, $\Pi_i(\mathbb{R}) < \infty$, $i = 1, 2$.

In what follows, we will assume that Assumption 1 holds.

Theorem 1. *There exists a unique global solution $X(t)$ to the system (4) for any initial value $X(0) = X_0 \in \mathbb{R}_+^2$, and $P\{X(t) \in \mathbb{R}_+^2\} = 1$.*

Proof. Let us consider the system of stochastic differential equations

$$\begin{aligned} dv_i(t) = & \left(r_i(t) - \frac{b_i(t) \exp\{v_i(t)\}}{K_i(t) + \exp\{v_{3-i}(t)\}} - c_i(t) \exp\{v_i(t)\} - \beta_i(t) \right) dt + \sigma_i(t) dw_i(t) + \\ & + \int_{\mathbb{R}} \ln(1 + \gamma_i(t, z)) \tilde{\nu}_1(dt, dz) + \int_{\mathbb{R}} \ln(1 + \delta_i(t, z)) \nu_2(dt, dz), \quad v_i(0) = \ln x_{i0}, \quad i = 1, 2. \end{aligned} \quad (5)$$

The coefficients of the system (5) are locally Lipschitz continuous. Therefore, for any initial value $(v_1(0), v_2(0))$ there exists a unique local solution $V(t) = (v_1(t), v_2(t))$ on $[0, \tau_e)$, where $\sup_{t < \tau_e} |V(t)| = +\infty$ (cf. Theorem 6, p. 246, (Gikhman, & Skorokhod, 1982)). So, from the Itô's formula, we derive that the process $X(t) = (\exp\{v_1(t)\}, \exp\{v_2(t)\})$ is a unique,

positive local solution to the equation (4). To show this solution is global, we need to show that $\tau_e = +\infty$ a.s. Let $n_0 \in \mathbb{N}$ be sufficiently large for $x_{i0} \in [1/n_0, n_0], i = 1, 2$.

For any $n \geq n_0$ we define the stopping time $\tau_n = \inf \left\{ t \in [0, \tau_e) : X(t) \notin \left(\frac{1}{n}, n \right) \times \left(\frac{1}{n}, n \right) \right\}$. It is easy to see that τ_n is increasing as $n \rightarrow +\infty$. Denote $\tau_\infty = \lim_{n \rightarrow \infty} \tau_n$, from which $\tau_\infty \leq \tau_e$ a.s. If we prove that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ a.s. and $X(t) \in \mathbb{R}_+^2$ a.s. for all $t \in [0, +\infty)$. So we need to show that $\tau_\infty = \infty$ a.s. If it is not true, there are constants $T > 0$ and $\varepsilon \in (0, 1)$, such that $P\{\tau_\infty < T\} > \varepsilon$. Hence, there is $n_1 \geq n_0$ such that

$$P\{\tau_n < T\} > \varepsilon, \quad \forall n \geq n_1. \quad (6)$$

For the non-negative function $V(X) = \sum_{i=1}^2 (x_i - 1 - \ln x_i)$, $X = (x_1, x_2)$, $x_i > 0, i = 1, 2$, by the Itô's formula we obtain

$$\begin{aligned} V(X(T \wedge \tau_n)) = V(X_0) + \sum_{i=1}^2 \left\{ \int_0^{T \wedge \tau_n} \left[(x_i(t) - 1) \left(r_i(t) - \frac{b_i(t)x_i(t)}{K_i(t) + x_{3-i}(t)} - c_i(t)x_i(t) \right) + \beta_i(t) + \right. \right. \\ \left. \left. + \int_{\mathbb{R}} \delta_i(t, z)x_i(t)\Pi_2(dz) \right] dt + \int_0^{T \wedge \tau_n} (x_i(t) - 1)\sigma_i(t)dw_i(t) + \int_0^{T \wedge \tau_n} \int_{\mathbb{R}} [\gamma_i(t, z)x_i(t^-) - \ln(1 + \gamma_i(t, z))]\tilde{\nu}_1(dt, dz) + \right. \\ \left. + \int_0^{T \wedge \tau_n} \int_{\mathbb{R}} [\delta_i(t, z)x_i(t^-) - \ln(1 + \delta_i(t, z))]\tilde{\nu}_2(dt, dz) \right\}. \end{aligned} \quad (7)$$

Under conditions of the theorem, there is a constant $L > 0$, such that

$$\begin{aligned} \sum_{i=1}^2 \left[(x_i - 1) \left(r_i(t) - \frac{b_i(t)x_i}{K_i(t) + x_{3-i}} - c_i(t)x_i \right) + \beta_i(t) + \int_{\mathbb{R}} \delta_i(t, z)x_i\Pi_2(dz) \right] \leq \\ \leq \sum_{i=1}^2 \left[\left(r_{\max} + c_{\max} + \frac{b_{\max}}{K_{\min}} + \delta_{\max}\Pi_2(\mathbb{R}) \right) x_i - c_{\min}x_i^2 + \beta_{\max} \right] \leq L, \end{aligned} \quad (8)$$

where $\delta_{\max} = \max_{i=1,2} \{\sup_{t \geq 0, z \in \mathbb{R}} \delta_i(t, z)\}$. From (7) and (8) we have

$$\begin{aligned} V(X(T \wedge \tau_n)) \leq V(X_0) + L(T \wedge \tau_n) + \sum_{i=1}^2 \left\{ \int_0^{T \wedge \tau_n} (x_i(t) - 1)\sigma_i(t)dw_i(t) + \right. \\ \left. + \int_0^{T \wedge \tau_n} \int_{\mathbb{R}} [\gamma_i(t, z)x_i(t^-) - \ln(1 + \gamma_i(t, z))]\tilde{\nu}_1(dt, dz) + \int_0^{T \wedge \tau_n} \int_{\mathbb{R}} [\delta_i(t, z)x_i(t^-) - \ln(1 + \delta_i(t, z))]\tilde{\nu}_2(dt, dz) \right\}. \end{aligned}$$

Whence taking expectations, yields

$$E[V(X(T \wedge \tau_n))] \leq V(X_0) + LT. \quad (9)$$

Set $\Omega_n = \{\tau_n \leq T\}$ for $n \geq n_1$. Then by (6), $P(\Omega_n) = P\{\tau_n \leq T\} > \varepsilon, \forall n \geq n_1$. Note that for every $\omega \in \Omega_n$ there is some i such that $x_i(\tau_n, \omega)$ equals either n or $1/n$. So $V(X(\tau_n)) \geq \min\{n - 1 - \ln n, \frac{1}{n} - 1 + \ln n\}$. It then follows from (9) that $V(X_0) + LT \geq E[\mathbf{1}_{\Omega_n} V(X(\tau_n))] \geq \varepsilon \min\{n - 1 - \ln n, \frac{1}{n} - 1 + \ln n\}$, where $\mathbf{1}_{\Omega_n}$ is the indicator function of Ω_n . Letting $n \rightarrow \infty$ leads to the contradiction $\infty > V(X_0) + LT = \infty$. This completes the proof of the theorem. $\square$

Lemma 1. Let $X(t) = (x_1(t), x_2(t))$ be a solution to the system (4). Then the density of the population $x_i(t), i = 1, 2$ has a property $\limsup_{t \rightarrow \infty} \frac{\ln x_i(t)}{t} \leq 0, i = 1, 2$ a.s.

Proof. By the Itô's formula we have for $i = 1, 2$

$$\begin{aligned} e^t \ln x_i(t) - \ln x_{i0} = \int_0^t e^s \left\{ \ln x_i(s) + r_i(s) - \frac{b_i(s)x_i(s)}{K_i(s) + x_{3-i}(s)} - \right. \\ \left. - c_i(s)x_i(s) - \frac{\sigma_i^2(s)}{2} + \int_{\mathbb{R}} [\ln(1 + \gamma_i(s, z)) - \gamma_i(s, z)]\Pi_1(dz) \right\} ds + \psi_i(t), \end{aligned} \quad (10)$$

where

$$\psi_i(t) = \int_0^t e^s \sigma_i(s)dw_i(s) + \int_0^t \int_{\mathbb{R}} e^s \ln(1 + \gamma_i(s, z))\tilde{\nu}_1(ds, dz) + \int_0^t \int_{\mathbb{R}} e^s \ln(1 + \delta_i(s, z))\nu_2(ds, dz), \quad i = 1, 2.$$

By virtue of the exponential inequality ((Borysenko, & Borysenko, 2019), Lemma 2.2) we have

$$P \left\{ \sup_{0 \leq t \leq T} \zeta_i(\mu, t) > \beta \right\} \leq e^{-\mu\beta}, \quad \forall 0 < \mu \leq 1, \quad \beta > 0, \quad i = 1, 2,$$

where

$$\begin{aligned} \zeta_i(\mu, t) = & \psi_i(t) - \frac{\mu}{2} \int_0^t e^{2s} \sigma_i^2(s) ds - \frac{1}{\mu} \int_0^t \int_{\mathbb{R}} \left[(1 + \gamma_i(s, z))^{\mu e^s} - 1 - \right. \\ & \left. - \mu e^s \ln(1 + \gamma_i(s, z)) \right] \Pi_1(dz) ds - \frac{1}{\mu} \int_0^t \int_{\mathbb{R}} \left[(1 + \delta_i(s, z))^{\mu e^s} - 1 \right] \Pi_2(dz) ds, \quad i = 1, 2. \end{aligned}$$

Choosing $T = k\tau, k \in \mathbb{N}, \tau > 0, \mu = e^{-k\tau}, \beta = \theta e^{k\tau} \ln k, \theta > 1$ we get

$$P \left\{ \sup_{0 \leq t \leq k\tau} \zeta_i(\mu, t) > \theta e^{k\tau} \ln k \right\} \leq \frac{1}{k^\theta}, \quad i = 1, 2.$$

By the Borel-Cantelli lemma for almost all $\omega \in \Omega$, there is a random integer $k_0(\omega)$, such that for $\forall k \geq k_0(\omega)$ and $0 \leq t \leq k\tau$ we have

$$\begin{aligned} \psi_i(t) \leq & \frac{1}{2e^{k\tau}} \int_0^t e^{2s} \sigma_i^2(s) ds + e^{k\tau} \int_0^t \int_{\mathbb{R}} \left[(1 + \gamma_i(s, z))^{e^{s-k\tau}} - 1 - e^{s-k\tau} \ln(1 + \gamma_i(s, z)) \right] \Pi_1(dz) ds + \\ & + e^{k\tau} \int_0^t \int_{\mathbb{R}} \left[(1 + \delta_i(s, z))^{e^{s-k\tau}} - 1 \right] \Pi_2(dz) ds + \theta e^{k\tau} \ln k, \quad i = 1, 2. \end{aligned} \quad (11)$$

Applying the inequality $x^r \leq 1 + r(x - 1)$, valid for all $x \geq 0, 0 \leq r \leq 1$, first for $x = 1 + \gamma_i(s, z), r = e^{s-k\tau}$, then for $x = 1 + \delta_i(s, z), r = e^{s-k\tau}$, we derive from (11) the estimates

$$\psi_i(t) \leq \frac{1}{2e^{k\tau}} \int_0^t e^{2s} \sigma_i^2(s) ds + \int_0^t \int_{\mathbb{R}} e^s [\gamma_i(s, z) - \ln(1 + \gamma_i(s, z))] \Pi_1(dz) ds + \int_0^t \int_{\mathbb{R}} e^s \delta_i(s, z) \Pi_2(dz) ds + \theta e^{k\tau} \ln k, \quad i = 1, 2. \quad (12)$$

So from (10) and (12), we get for $i = 1, 2$

$$\begin{aligned} e^t \ln x_i(t) \leq & \ln x_{i0} + \int_0^t e^s \left\{ \ln x_i(s) + r_i(s) - \frac{b_i(s)x_i(s)}{K_i(s) + x_{3-i}(s)} - c_i(s)x_i(s) - \frac{\sigma_i^2(s)}{2} (1 - e^{s-k\tau}) + \right. \\ & \left. + \int_{\mathbb{R}} \delta_i(s, z) \Pi_2(dz) \right\} ds + \theta e^{k\tau} \ln k \leq \ln x_{i0} + \int_0^t e^s [\ln x_i(s) - c_i \inf x_i(s) + L_i] ds + \theta e^{k\tau} \ln k \leq \\ & \leq \ln x_{i0} + L(e^t - 1) + \theta e^{k\tau} \ln k, \quad \forall k \geq k_0(\omega), \quad 0 \leq t \leq k\tau, \end{aligned}$$

for some constant $L > 0$, where $L_i = r_i \sup + \sup_{t \geq 0} \int_{\mathbb{R}} \delta_2(t, z) \Pi_2(dz)$. So for any $(k-1)\tau \leq t \leq k\tau, \forall k \geq k_0(\omega)$ we have

$$\frac{\ln x_i(t)}{\ln t} \leq e^{-t} \frac{\ln x_{i0}}{\ln t} + \frac{L}{\ln t} (1 - e^{-t}) + \frac{\theta e^{k\tau} \ln k}{e^{(k-1)\tau} \ln(k-1)\tau}, \quad i = 1, 2 \quad \text{a.s.}$$

Therefore

$$\limsup_{t \rightarrow \infty} \frac{\ln x_i(t)}{\ln t} \leq \theta e^\tau, \quad i = 1, 2, \quad \forall \theta > 1, \quad \forall \tau > 0, \quad \text{a.s.}$$

If $\theta \downarrow 1, \tau \downarrow 0$, then we obtain

$$\limsup_{t \rightarrow \infty} \frac{\ln x_i(t)}{\ln t} \leq 1, \quad i = 1, 2 \quad \text{a.s.}$$

So $\limsup_{t \rightarrow \infty} \frac{\ln x_i(t)}{t} \leq 0, \quad i = 1, 2 \quad \text{a.s.}$ □

Lemma 2. Let $p > 0$. Then for any initial value $x_{i0} > 0, i = 1, 2$ we have

$$\limsup_{t \rightarrow \infty} E[x_i^p(t)] \leq L_i(p), \quad i = 1, 2,$$

where $L_i(p) > 0, i = 1, 2$ are some constants depending on p .

Proof. Let τ_n be the stopping time defined in Theorem 1. Applying the Itô's formula to the process $V(t, x_i(t)) = e^t x_i^p(t)$,

$i = 1, 2, p > 0$, we obtain for $i = 1, 2$

$$\begin{aligned} V(t \wedge \tau_n, x_i(t \wedge \tau_n)) &= x_{i0}^p + \int_0^{t \wedge \tau_n} e^s x_i^p(s) \left\{ 1 + p \left(r_i(s) - \frac{b_i(s)x_i(s)}{K_i(s) + x_{3-i}(s)} - c_i(s)x_i(s) \right) + \frac{p(p-1)\sigma_i^2(s)}{2} + \right. \\ &+ \int_{\mathbb{R}} [(1 + \gamma_i(s, z))^p - 1 - p\gamma_i(s, z)] \Pi_1(dz) + \int_{\mathbb{R}} [(1 + \delta_i(s, z))^p - 1] \Pi_2(dz) \Big\} ds + \int_0^{t \wedge \tau_n} p e^s x_i^p(s) \sigma_i(s) dw_i(s) + \\ &+ \int_0^{t \wedge \tau_n} \int_{\mathbb{R}} e^s x_i^p(s^-) [(1 + \gamma_i(s, z))^p - 1] \tilde{\nu}_1(ds, dz) + \int_0^{t \wedge \tau_n} \int_{\mathbb{R}} e^s x_i^p(s^-) [(1 + \delta_i(s, z))^p - 1] \tilde{\nu}_2(ds, dz). \end{aligned} \quad (13)$$

Under Assumption 1 there are constants $L_i(p) > 0, i = 1, 2$, such that

$$\begin{aligned} e^s x_i^p(s) \left\{ 1 + p \left(r_i(s) - \frac{b_i(s)x_i(s)}{K_i(s) + x_{3-i}(s)} - c_i(s)x_i(s) \right) + \frac{p(p-1)\sigma_i^2(s)}{2} + \right. \\ \left. + \int_{\mathbb{R}} [(1 + \gamma_i(s, z))^p - 1 - p\gamma_i(s, z)] \Pi_1(dz) + \int_{\mathbb{R}} [(1 + \delta_i(s, z))^p - 1] \Pi_2(dz) \right\} \leq \\ \leq e^s \left[x_i^p(s) \left(1 + p r_i \sup + \frac{p|p-1|}{2} \sigma_{i \sup}^2 + J_i \sup \right) - p c_{i \inf} x_i^{p+1}(s) \right] \leq e^s L_i(p), \end{aligned} \quad (14)$$

where

$$J_i \sup = \sup_{s \geq 0} \left\{ \int_{\mathbb{R}} [(1 + \gamma_i(s, z))^p - 1 - p\gamma_i(s, z)] \Pi_1(dz) + \int_{\mathbb{R}} [(1 + \delta_i(s, z))^p - 1] \Pi_2(dz) \right\}.$$

From (13) and (14), taking the expectation, we obtain

$$\mathbb{E}[V(t \wedge \tau_n, x_i(t \wedge \tau_n))] \leq x_{i0}^p + L_i(p)e^t, \quad i = 1, 2.$$

If $n \rightarrow \infty$, then we get

$$e^t \mathbb{E}[x_i^p(t)] \leq x_{i0}^p + L_i(p)e^t, \quad i = 1, 2.$$

Hence $\limsup_{t \rightarrow \infty} \mathbb{E}[x_i^p(t)] \leq L_i(p), \quad i = 1, 2.$

□

Lemma 3. If $\min\{(r_1 - \beta_1)_{\inf}, (r_2 - \beta_2)_{\inf}\} > 0$, then for any initial value $x_{i0} > 0, i = 1, 2$ we have

$$\limsup_{t \rightarrow \infty} \mathbb{E} \left[\left(\frac{1}{x_i(t)} \right)^\theta \right] \leq L_i(\theta), \quad 0 < \theta < 1, \quad i = 1, 2, \quad (15)$$

where $L_i(\theta) > 0, i = 1, 2$ are some constants depending on $0 < \theta < 1$.

Proof. For the process $U_i(t) = 1/x_i(t), i = 1, 2$ by the Itô's formula we have

$$\begin{aligned} U_i(t) &= U_i(0) + \int_0^t U_i(s) \left[-r_i(s) + \frac{b_i(s)x_i(s)}{K_i(s) + x_{3-i}(s)} + c_i(s)x_i(s) + \sigma_i^2(s) + \int_{\mathbb{R}} \frac{\gamma_i^2(s, z)}{1 + \gamma_i(s, z)} \Pi_1(dz) \right] ds - \\ &- \int_0^t U_i(s) \sigma_i(s) dw_i(s) - \int_0^t \int_{\mathbb{R}} U_i(s^-) \frac{\gamma_i(s, z)}{1 + \gamma_i(s, z)} \tilde{\nu}_1(ds, dz) - \int_0^t \int_{\mathbb{R}} U_i(s^-) \frac{\delta_i(s, z)}{1 + \delta_i(s, z)} \nu_2(ds, dz). \end{aligned}$$

Then by the Itô's formula we derive for $0 < \theta < 1, i = 1, 2$

$$\begin{aligned} (1 + U_i(t))^\theta &= (1 + U_i(0))^\theta + \int_0^t \theta (1 + U_i(s))^{\theta-2} \left\{ (1 + U_i(s)) U_i(s) \left[-r_i(s) + \frac{b_i(s)x_i(s)}{K_i(s) + x_{3-i}(s)} + c_i(s)x_i(s) + \right. \right. \\ &+ \sigma_i^2(s) + \int_{\mathbb{R}} \frac{\gamma_i^2(s, z)}{1 + \gamma_i(s, z)} \Pi_1(dz) \Big] + \frac{\theta-1}{2} U_i^2(s) \sigma_i^2(s) + \frac{1}{\theta} \int_{\mathbb{R}} \left[(1 + U_i(s))^2 \left(\frac{1 + U_i(s) + \gamma_i(s, z)}{(1 + \gamma_i(s, z))(1 + U_i(s))} \right)^\theta - 1 \right] + \\ &+ \theta (1 + U_i(s)) \frac{U_i(s) \gamma_i(s, z)}{1 + \gamma_i(s, z)} \Big] \Pi_1(dz) + \frac{1}{\theta} \int_{\mathbb{R}} (1 + U_i(s))^2 \left[\left(\frac{1 + U_i(s) + \delta_i(s, z)}{(1 + \delta_i(s, z))(1 + U_i(s))} \right)^\theta - 1 \right] \Pi_2(dz) \Big\} ds - \\ &- \int_0^t \theta (1 + U_i(s))^{\theta-1} U_i(s) \sigma_i(s) dw_i(s) + \int_0^t \int_{\mathbb{R}} \left[\left(1 + \frac{U_i(s^-)}{1 + \gamma_i(s, z)} \right)^\theta - (1 + U_i(s^-))^\theta \right] \tilde{\nu}_1(ds, dz) + \end{aligned} \quad (16)$$

$$+\int_0^t \int_{\mathbb{R}} \left[\left(1 + \frac{U_i(s^-)}{1+\delta_i(s,z)} \right)^\theta - (1+U_i(s^-))^\theta \right] \tilde{\nu}_2(ds, dz) = (1+U_i(0))^\theta + \int_0^t \theta (1+U_i(s))^{\theta-2} J_i(s) ds + \sum_{j=1}^3 I_{j, stoch}(t),$$

where $I_{j, stoch}(t), j = \overline{1, 3}$ are corresponding stochastic integrals in (16). Under Assumption 1 there exist constants $|L_{i,1}(\theta)| < \infty, |L_{i,2}(\theta)| < \infty, i = 1, 2$ such, that for the process $J_i(t)$ we have the estimate

$$\begin{aligned} J_i(t) \leq & (1+U_i(t))U_i(t) \left[-r_i(t) + U_i^{-1}(t) \left(\frac{b_i(t)}{K_i(t)} + c_i(t) \right) + \sigma_i^2(t) + \int_{\mathbb{R}} \gamma_i(t, z) \Pi_1(dz) \right] + \\ & + \frac{\theta-1}{2} U_i^2(t) \sigma_i^2(t) + \frac{1}{\theta} \int_{\mathbb{R}} (1+U_i(t))^2 \left[\left(\frac{1}{1+\gamma_i(t, z)} + \frac{1}{1+U_i(t)} \right)^\theta - 1 \right] \Pi_1(dz) + \\ & + \frac{1}{\theta} \int_{\mathbb{R}} (1+U_i(t))^2 \left[\left(\frac{1}{1+\delta_i(t, z)} + \frac{1}{1+U_i(t)} \right)^\theta - 1 \right] \Pi_2(dz) \leq U_i^2(t) \left[-r_i(t) + \frac{\sigma_i^2(t)}{2} + \right. \\ & \left. + \int_{\mathbb{R}} \gamma_i(t, z) \Pi_1(dz) + \frac{\theta}{2} \sigma_i^2(t) + \frac{1}{\theta} \int_{\mathbb{R}} [(1+\gamma_i(t, z))^{-\theta} - 1] \Pi_1(dz) + \frac{1}{\theta} \int_{\mathbb{R}} [(1+\delta_i(t, z))^{-\theta} - 1] \Pi_2(dz) \right] + \\ & + L_{i,1}(\theta)U_i(t) + L_{i,2}(\theta) = -L_{i,0}(t, \theta)U_i^2(t) + L_{i,1}(\theta)U_i(t) + L_{i,2}(\theta). \end{aligned}$$

Here we used the inequality $(x+y)^\theta \leq x^\theta + \theta x^{\theta-1}y$, valid for all $0 < \theta < 1, x, y > 0$, in the first integral term for $x = (1+\gamma_i(t, z))^{-1}, y = (1+U_i(t))^{-1}$, and then in the second integral term for $x = (1+\delta_i(t, z))^{-1}, y = (1+U_i(t))^{-1}$. Due to

$$\begin{aligned} \lim_{\theta \rightarrow 0+} \left[\frac{\theta}{2} \sigma_i^2(t) + \frac{1}{\theta} \int_{\mathbb{R}} [(1+\gamma_i(t, z))^{-\theta} - 1] \Pi_1(dz) + \frac{1}{\theta} \int_{\mathbb{R}} [(1+\delta_i(t, z))^{-\theta} - 1] \Pi_2(dz) + \right. \\ \left. + \int_{\mathbb{R}} \ln(1+\gamma_i(t, z)) \Pi_1(dz) + \int_{\mathbb{R}} \ln(1+\delta_i(t, z)) \Pi_2(dz) \right] = \lim_{\theta \rightarrow 0+} \Delta_i(\theta, t) = 0, \end{aligned}$$

and the condition $\min_{i=1,2} (r_i - \beta_i)_{\inf} > 0$ we can choose a sufficiently small $0 < \theta < 1$ so that

$$L_{i,0}(\theta) = \inf_{t \geq 0} L_{i,0}(t, \theta) = \inf_{t \geq 0} [r_i(t) - \beta_i(t) - \Delta_i(\theta, t)] > 0$$

is satisfied. So from (16) and the estimate for $J_i(t)$ we derive

$$\begin{aligned} d \left[(1+U_i(t))^\theta \right] \leq & \theta (1+U_i(t))^{\theta-2} [-L_{i,0}(\theta)U_i^2(t) + L_{i,1}(\theta)U_i(t) + L_{i,2}(\theta)] dt - \theta (1+U_i(t))^{\theta-1} U_i(t) \sigma_i(t) dw_i(t) + \\ & + \int_{\mathbb{R}} \left[\left(1 + \frac{U_i(t^-)}{1+\gamma_i(t, z)} \right)^\theta - (1+U_i(t^-))^\theta \right] \tilde{\nu}_1(dt, dz) + \int_{\mathbb{R}} \left[\left(1 + \frac{U_i(t^-)}{1+\delta_i(t, z)} \right)^\theta - (1+U_i(t^-))^\theta \right] \tilde{\nu}_2(dt, dz). \end{aligned} \quad (17)$$

By the Itô's formula and (17) we have

$$\begin{aligned} d \left[e^{\lambda t} (1+U_i(t))^\theta \right] = & \lambda e^{\lambda t} (1+U_i(t))^\theta dt + e^{\lambda t} d \left[(1+U_i(t))^\theta \right] \leq e^{\lambda t} \theta (1+U_i(t))^{\theta-2} \left[- \left(L_{i,0}(\theta) - \frac{\lambda}{\theta} \right) U_i^2(t) + \right. \\ & + \left(L_{i,1}(\theta) + \frac{2\lambda}{\theta} \right) U_i(t) + L_{i,2}(\theta) + \frac{\lambda}{\theta} \right] dt - \theta e^{\lambda t} (1+U_i(t))^{\theta-1} U_i(t) \sigma_i(t) dw_i(t) + \\ & + e^{\lambda t} \int_{\mathbb{R}} \left[\left(1 + \frac{U_i(t^-)}{1+\gamma_i(t, z)} \right)^\theta - (1+U_i(t^-))^\theta \right] \tilde{\nu}_1(dt, dz) + e^{\lambda t} \int_{\mathbb{R}} \left[\left(1 + \frac{U_i(t^-)}{1+\delta_i(t, z)} \right)^\theta - (1+U_i(t^-))^\theta \right] \tilde{\nu}_2(dt, dz). \end{aligned} \quad (18)$$

Let us choose $\lambda = \lambda(\theta) > 0$ such that $\min_{i=1,2} L_{i,0}(\theta) - \lambda/\theta > 0$. Then there is a constants $L_i(\theta) > 0, i = 1, 2$, such that for $i = 1, 2$ we have

$$(1+U_i(t))^{\theta-2} \left[- \left(L_{i,0}(\theta) - \frac{\lambda}{\theta} \right) U_i^2(t) + \left(L_{i,1}(\theta) + \frac{2\lambda}{\theta} \right) U_i(t) + L_{i,2}(\theta) + \frac{\lambda}{\theta} \right] \leq L_i(\theta). \quad (19)$$

Let τ_n be the stopping time defined in the Theorem 1. Then by integrating (18) from 0 to $t \wedge \tau_n$, using (19) and taking the expectation we obtain

$$\mathbb{E} \left[e^{\lambda(t \wedge \tau_n)} (1+U_i(t \wedge \tau_n))^\theta \right] \leq \left(1 + \frac{1}{x_{i0}} \right)^\theta + \frac{\theta}{\lambda} L_i(\theta) (e^{\lambda t} - 1).$$

Letting $n \rightarrow \infty$ leads to the estimate

$$e^t \mathbb{E} \left[(1+U_i(t))^\theta \right] \leq \left(1 + \frac{1}{x_{i0}} \right)^\theta + \frac{\theta}{\lambda} L_i(\theta) (e^{\lambda t} - 1). \quad (20)$$

From (20) we obtain

$$\limsup_{t \rightarrow \infty} E \left[\left(\frac{1}{x_i(t)} \right)^\theta \right] = \limsup_{t \rightarrow \infty} E \left[U_i^\theta(t) \right] \leq \limsup_{t \rightarrow \infty} E \left[(1 + U_i(t))^\theta \right] \leq \frac{\theta}{\lambda(\theta)} L_i(\theta),$$

and this implies (15). $\square$

2. The long time behaviour

Definition 1. ((Li, & Mao, 2009)) The solution $X(t)$ to the system (4) will be said stochastically ultimately bounded, if for any $\varepsilon \in (0, 1)$, there is a positive constant $\chi = \chi(\varepsilon) > 0$, such that for any initial value $X_0 \in \mathbb{R}_+^2$, the solution to the system (4) has the property that

$$\limsup_{t \rightarrow \infty} P \{ |X(t)| > \chi \} < \varepsilon.$$

In what follows in this section we will assume that Assumption 1 holds.

Theorem 2. The solution $X(t)$ to the system (4) is stochastically ultimately bounded for any initial value $X_0 \in \mathbb{R}_+^2$.

Proof. From Lemma 2 we have an estimate

$$\limsup_{t \rightarrow \infty} E[x_i(t)] \leq L_i, \quad i = 1, 2. \quad (21)$$

For $X = (x_1, x_2) \in \mathbb{R}_+^2$ we have $|X| \leq x_1 + x_2$, therefore, from (21) $\limsup_{t \rightarrow \infty} E[|X(t)|] \leq L = L_1 + L_2$. Let $\chi > L/\varepsilon$, $\forall \varepsilon \in (0, 1)$. Then applying the Chebyshev inequality yields

$$\limsup_{t \rightarrow \infty} P\{|X(t)| > \chi\} \leq \frac{1}{\chi} \limsup_{t \rightarrow \infty} E|X(t)| \leq \frac{L}{\chi} < \varepsilon.$$

$\square$

Definition 2. ((Li, Gao, & Wang, 2016)) The solution $X(t)$ to the equation (4) is said to be stochastically permanent if for any $\varepsilon > 0$, there are positive constants $H = H(\varepsilon)$, $h = h(\varepsilon)$ such that for $i = 1, 2$

$$\liminf_{t \rightarrow \infty} P\{x_i(t) \leq H\} \geq 1 - \varepsilon, \quad \liminf_{t \rightarrow \infty} P\{x_i(t) \geq h\} \geq 1 - \varepsilon,$$

for any initial value $X_0 \in \mathbb{R}_+^2$.

Theorem 3. Under the conditions of Lemma 3, the solution $X(t)$ to the system (4) is stochastically permanent for any initial value $X_0 \in \mathbb{R}_+^2$.

Proof. From Lemma 2 we have an estimate

$$\limsup_{t \rightarrow \infty} E[x_i(t)] \leq L, \quad i = 1, 2.$$

Thus for any given $\varepsilon > 0$, let $H = L/\varepsilon$, by Chebyshev's inequality, we can derive that

$$\limsup_{t \rightarrow \infty} P\{x_i(t) > H\} \leq \frac{1}{H} \limsup_{t \rightarrow \infty} E[x_i(t)] \leq \varepsilon.$$

Consequently $\liminf_{t \rightarrow \infty} P\{x_i(t) \leq H\} \geq 1 - \varepsilon$, $i = 1, 2$.

From Lemma 3 we have the estimate

$$\limsup_{t \rightarrow \infty} E \left[\left(\frac{1}{x_i(t)} \right)^\theta \right] \leq L(\theta), \quad 0 < \theta < 1.$$

For any given $\varepsilon > 0$, let $h = (\varepsilon/L(\theta))^{1/\theta}$, then by Chebyshev's inequality, we have

$$\limsup_{t \rightarrow \infty} P\{x_i(t) < h\} \leq \limsup_{t \rightarrow \infty} P \left\{ \left(\frac{1}{x_i(t)} \right)^\theta > h^{-\theta} \right\} \leq h^\theta \limsup_{t \rightarrow \infty} E \left[\left(\frac{1}{x_i(t)} \right)^\theta \right] \leq \varepsilon.$$

Consequently $\liminf_{t \rightarrow \infty} P\{x_i(t) \geq h\} \geq 1 - \varepsilon$. $\square$

Theorem 4. If for $i = 1, 2$ we have

$$\bar{p}_i^* = \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t p_i(s) ds < 0, \quad \text{where } p_i(s) = r_i(s) - \beta_i(s),$$

then the solution $X(t)$ to the system (4) with initial condition $X_0 \in \mathbb{R}_+^2$ will be extinct: $\lim_{t \rightarrow \infty} x_i(t) = 0$, $i = 1, 2$ a.s.

Proof. By the Itô's formula, we have for $i = 1, 2$

$$\ln x_i(t) = \ln x_{i0} + \int_0^t \left[r_i(s) - \frac{b_i(s)x_i(s)}{K_i(s) + x_{3-i}(s)} - c_i(s)x_i(s) - \beta_i(s) \right] ds + M_i(t) \leq \ln x_{i0} + \int_0^t p_i(s)ds + M_i(t), \quad (22)$$

where the martingale

$$M_i(t) = \int_0^t \sigma_i(s)dw_i(s) + \int_0^t \int_{\mathbb{R}} \ln(1 + \gamma_i(s, z))\tilde{\nu}_1(ds, dz) + \int_0^t \int_{\mathbb{R}} \ln(1 + \delta_i(s, z))\tilde{\nu}_2(ds, dz), \quad i = 1, 2, \quad (23)$$

has quadratic variation (Meyer's angle bracket process)

$$\langle M_i, M_i \rangle(t) = \int_0^t \sigma_i^2(s)ds + \int_0^t \int_{\mathbb{R}} \ln^2(1 + \gamma_i(s, z))\Pi_1(dz)ds + \int_0^t \int_{\mathbb{R}} \ln^2(1 + \delta_i(s, z))\Pi_2(dz)ds \leq Kt, \quad i = 1, 2.$$

Then the strong law of large numbers for local martingales (Lipster, 1980) yields $\lim_{t \rightarrow \infty} M_i(t)/t = 0, i = 1, 2$ a.s. Therefore, from (22) we obtain

$$\limsup_{t \rightarrow \infty} \frac{\ln x_i(t)}{t} \leq \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t p_i(s)ds < 0, \quad \text{a.s.}$$

So $\lim_{t \rightarrow \infty} x_i(t) = 0, i = 1, 2$ a.s. □

Theorem 5. If $\bar{p}_i^* = 0, i = 1, 2$, then the solution $X(t)$ to the system (4) with initial condition $X_0 \in \mathbb{R}_+^2$ will be non-persistent in the mean:

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t x_i(s)ds = 0 \quad \text{a.s., } i = 1, 2.$$

Proof. From the first equality in (22) we have

$$\ln x_i(t) \leq \ln x_{i0} + \int_0^t p_i(s)ds - c_i \inf \int_0^t x_i(s)ds + M_i(t), \quad i = 1, 2, \quad (24)$$

where martingale $M_i(t)$ is defined in (23). From the definition of $\bar{p}_i^*$ and the strong law of large numbers for $M_i(t)$ it follows, that $\forall \varepsilon > 0, \exists t_0 \geq 0$, and $\exists \Omega_\varepsilon \subset \Omega, P(\Omega_\varepsilon) \geq 1 - \varepsilon$ such that

$$\frac{1}{t} \int_0^t p_i(s)ds \leq \bar{p}_i^* + \frac{\varepsilon}{2}, \quad \frac{M_i(t)}{t} \leq \frac{\varepsilon}{2}, \quad \forall t \geq t_0, \omega \in \Omega_\varepsilon.$$

So, from (24) and condition $\bar{p}_i^* = 0, i = 1, 2$ we derive

$$\ln x_i(t) - \ln x_{i0} \leq t(\bar{p}_i^* + \varepsilon) - c_i \inf \int_0^t x_i(s)ds = t\varepsilon - c_i \inf \int_0^t x_i(s)ds, \quad i = 1, 2, \quad \forall t \geq t_0, \omega \in \Omega_\varepsilon. \quad (25)$$

Let $y_i(t) = \int_0^t x_i(s)ds, i = 1, 2$, then from (25) we have

$$\ln \left(\frac{dy_i(t)}{dt} \right) \leq \varepsilon t - c_i \inf y_i(t) + \ln x_{i0}, \quad i = 1, 2, \quad \forall t \geq t_0, \omega \in \Omega_\varepsilon.$$

Therefore

$$e^{c_i \inf y_i(t)} \frac{dy_i(t)}{dt} \leq x_{i0} e^{\varepsilon t}, \quad i = 1, 2, \quad \forall t \geq t_0, \omega \in \Omega_\varepsilon.$$

By integrating the last inequality from t_0 to t we obtain

$$e^{c_i \inf y_i(t)} \leq \frac{c_i \inf x_{i0}}{\varepsilon} (e^{\varepsilon t} - e^{\varepsilon t_0}) + e^{c_i \inf y_i(t_0)}, \quad i = 1, 2, \quad \forall t \geq t_0, \omega \in \Omega_\varepsilon.$$

So

$$y_i(t) \leq \frac{1}{c_i \inf} \ln \left[e^{c_i \inf y_i(t_0)} + \frac{c_i \inf x_{i0}}{\varepsilon} (e^{\varepsilon t} - e^{\varepsilon t_0}) \right], \quad i = 1, 2, \quad \forall t \geq t_0, \omega \in \Omega_\varepsilon,$$

and therefore

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t x_i(s)ds \leq \frac{\varepsilon}{c_i \inf}, \quad i = 1, 2, \quad \forall \omega \in \Omega_\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary and $x_i(t) > 0, i = 1, 2$ a.s., we have $\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t x_i(s) ds = 0, i = 1, 2$ a.s. $\square$

Theorem 6. If $\bar{p}_i^* > 0, i = 1, 2$, then the solution $X(t)$ to the system (4) with initial condition $X_0 \in \mathbb{R}_+^2$ will be weakly persistent in the mean:

$$\bar{x}_i^* = \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t x_i(s) ds > 0, i = 1, 2, \quad \text{a.s.}$$

Proof. If the assertion of the theorem is not true, then $P\{\omega \in \Omega \mid \bar{x}_2^*(\omega) = 0\} > 0$. From the first equality in (22) we get

$$\frac{\ln x_i(t)}{t} \geq \frac{\ln x_{i0}}{t} + \frac{1}{t} \int_0^t p_i(s) ds - \left(\frac{b_i \sup}{K_i \inf} + c_i \sup \right) \frac{1}{t} \int_0^t x_i(s) ds + \frac{M_i(t)}{t}, i = 1, 2, \quad (26)$$

where martingale $M_i(t), i = 1, 2$ is defined in (23). For $\forall \omega \in \{\omega \in \Omega \mid \bar{x}_2^* = 0\}$ in virtue of the strong law of large numbers for martingale $M_i(t), i = 1, 2$ we have

$$\limsup_{t \rightarrow \infty} \frac{\ln x_i(t)}{t} \geq \bar{p}_i^* > 0, i = 1, 2.$$

Therefore

$$P\left\{\omega \in \Omega : \limsup_{t \rightarrow \infty} \frac{\ln x_i(t)}{t} > 0\right\} > 0, i = 1, 2.$$

But from Lemma 1 we have

$$P\left\{\omega \in \Omega : \limsup_{t \rightarrow \infty} \frac{\ln x_i(t)}{t} \leq 0\right\} = 1, i = 1, 2.$$

So we obtain a contradiction. $\square$

Theorem 7. If $\bar{p}_{i*} = \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t p_i(s) ds > 0, i = 1, 2$, where $p_i(t) = r_i(t) - \beta_i(t), i = 1, 2$, then

$$\bar{x}_{i*} = \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t x_i(s) ds \geq \frac{\bar{p}_{i*}}{m_i}, \text{ a.s., where } m_i = \frac{b_i \inf}{K_i \inf} + c_i \sup, i = 1, 2,$$

Therefore, the solution $X(t) = (x_1(t), x_2(t))$ of system (4) with initial value $X_0 \in \mathbb{R}_+^2$ will be strongly persistent in the mean: $\bar{x}_{i*} > 0$ almost surely, $i = 1, 2$.

Proof. From (26) we get

$$\ln x_i(t) \geq \ln x_{i0} + \int_0^t p_i(s) ds - m_i \int_0^t x_i(s) ds + M_i(t), i = 1, 2, \quad (27)$$

where martingales $M_i(t), i = 1, 2$ are defined in (23). From the definition of $\bar{p}_{i*}, i = 1, 2$ and the strong law of large numbers for $M_i(t), i = 1, 2$ it follows that $\forall \varepsilon > 0, \exists t_0 \geq 0$, and $\exists \Omega_\varepsilon \subset \Omega$, such that $P(\Omega_\varepsilon) \geq 1 - \varepsilon$,

$$\frac{1}{t} \int_0^t p_i(s) ds \geq \bar{p}_{i*} - \frac{\varepsilon}{2}, \frac{M_i(t)}{t} \geq -\frac{\varepsilon}{2}, i = 1, 2, \forall t \geq t_0, \omega \in \Omega_\varepsilon.$$

Hence, from (27) we have

$$\ln x_i(t) \geq \ln x_{i0} + t(\bar{p}_{i*} - \varepsilon) - m_i \int_0^t x_i(s) ds, i = 1, 2, \forall t \geq t_0, \omega \in \Omega_\varepsilon.$$

Applying the same arguments for the processes $y_i(t) = \int_0^t x_i(s) ds, i = 1, 2$ as in the proof of Theorem 5, we derive the estimate

$$y_i(t) \geq \frac{1}{m_i} \ln \left[e^{m_i y_i(t_0)} + \frac{m_i x_{i0}}{\bar{p}_{i*} - \varepsilon} \left(e^{(\bar{p}_{i*} - \varepsilon)t} - e^{(\bar{p}_{i*} - \varepsilon)t_0} \right) \right], i = 1, 2, \forall t \geq t_0, \omega \in \Omega_\varepsilon.$$

So

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t x_i(s) ds \geq \frac{\bar{p}_{i*} - \varepsilon}{m_i}, i = 1, 2, \omega \in \Omega_\varepsilon.$$

Using the arbitrariness of $\varepsilon > 0$, we complete the proof of the theorem. $\square$

Discussion and conclusions

In this paper, we consider the non-autonomous stochastic mutualism model driven by the system of stochastic differential equations with white noise, centered and non-centered Poisson noises. So, we take into account not only "small" jumps,

corresponding to the centered Poisson measure, but also the “large” jumps, corresponding to the non-centered Poisson measure. For the considered system, sufficient criteria for the existence and uniqueness of a global positive solution are obtained. We derive sufficient conditions of stochastic ultimate boundedness, stochastic permanence, non-persistence in the mean, weak and strong persistence in the mean, and extinction of the population densities in the considered stochastic mutualism model. The results strictly generalize existing results, so they are meaningful and important.

Authors' contribution: Oleksandr Borysenko – conceptualization, writing, and editing; Olga Borysenko – methodology, writing, viewing, and editing.

Acknowledgments. The authors would like to thank the anonymous referee who provided useful and detailed comments on an earlier version of the manuscript.

Sources of funding. The study was conducted mostly at the authors' own expense. Some financial support was partially provided by Taras Shevchenko National University of Kyiv.

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Отримано редакцією журналу / Received: 29.03.25

Прорецензовано / Revised: 20.02.26

Схвалено до друку / Accepted: 16.04.26

Опубліковано / Published: 05.06.26

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ДИНАМІКА НЕАВТОНОМНОЇ СТОХАСТИЧНОЇ МОДЕЛІ МУТУАЛІЗМУ ЗІ СТРИБКАМИ

У природі ми можемо знайти багато прикладів взаємодії двох або більшої кількості популяцій, за яких вигоди одержує кожна із популяцій. Моделі мутуалізму виникають тоді, коли одна популяція надає певні вигоди іншій в обмін на вигоди для себе. Прикладом можуть слугувати популяція мурах і популяція попелюхи. Математичні детерміновані моделі широко використовуються під час вивчення динаміки таких популяційних систем. У реальному житті на популяційні системи часто впливають випадкові збурення, тому доцільно використовувати стохастичні моделі популяційної динаміки. Якщо ми хочемо враховувати такі раптові збурення середовища, як епідемії, пожежі, землетруси, тоді нам доцільно ввести центрований і нецентрований пуассонівські шуми у популяційну модель. Отже, ми будемо враховувати не тільки “малі” стрибки, що відповідають центрованому пуассонівському шуму, але і “великі” стрибки, що відповідають нецентрованому пуассонівському шуму. У роботі доведено існування і єдиність глобального додатного розв’язку системи стохастичних диференціальних рівнянь, що описує неавтономну модель мутуалізму, збурену “білим” шумом, центрованим і нецентрованим пуассонівськими шумами. Одержано достатні умови граничної стохастичної обмеженості, стохастичної перманентності, вимирання, неживання у середньому, слабкого і сильного виживання у середньому для розв’язку системи стохастичних диференціальних рівнянь, що описує відповідну модель мутуалізму.

Ключові слова: стохастична модель мутуалізму, глобальний розв’язок, гранична стохастична обмеженість, стохастична перманентність, вимирання, неживання у середньому, слабке і сильне виживання у середньому.

Автори заявляють про відсутність конфлікту інтересів. Спонсори не брали участі в розробленні дослідження; у зборі, аналізі чи інтерпретації даних; у написанні рукопису; в рішенні про публікацію результатів.

The authors declare no conflicts of interest. The funders had no role in the design of the study; in the collection, analyses or interpretation of data; in the writing of the manuscript; in the decision to publish the results.