УДК 519.21

<https://doi.org/10.17721/1812-5409.2020/1-2.2>

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Властивості φ -субгауссових випадкових процесів, пов'язаних із рівнянням теплопровідності з випадковими початковими умовами

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Properties of φ -sub-Gaussian stochastic processes related to the heat equation with random initial conditions

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У роботі досліджуються властивості траєкторій випадкових процесів, що задають розв'язки (в $L_2(\Omega)$) для рівняння теплопровідності з φ -субгауссовими стаціонарними випадковими умовами. Основні результати роботи – це оцінки для розподілів супремумів вказаних процесів на обмежених та необмежених множинах.

Ключові слова: φ -субгауссові процеси, рівняння теплопровідності, випадкові початкові умови, розподіл супремуму

In this paper, there are studied sample paths properties of stochastic processes representing solutions (in $L_2(\Omega)$ sense) of the heat equation with random initial conditions given by φ -sub-Gaussian stationary processes. The main results are the bounds for the distributions of the suprema for such stochastic processes considered over bounded and unbounded domains.

Key Words: φ -sub-Gaussian processes, heat equation, random initial condition, distribution of supremum

1 Introduction

In this paper we investigate sample paths properties of stochastic processes representing solutions of the heat equation with random initial conditions given by φ -sub-Gaussian stationary processes. The general theory of φ -sub-Gaussian processes is presented in the classical monograph [2] and its further development can be found in numerous recent studies. Partial differential equations with random initial conditions have been intensively studied in the literature (from different points of view), starting from the papers by J. Kampé de Fériet (1955) and M. Rosenblatt (1967) who introduced rigorous probabilistic tools in this area. In particular, in [8, 9] solutions to PDE subject to random initial conditions were investigated by means of Fourier methods, representations

of solutions by uniformly convergent series and their approximations in different functional spaces were developed. The present paper is most closely related to the papers [1, 3, 4, 6].

We obtain the bounds for distribution of supremum for the processes related to the heat equations with φ -sub-Gaussian initial conditions. These results generalize the corresponding results from papers [3, 4] where the cases of Gaussian and sub-Gaussian initial conditions were considered. We also investigate the rate of growth of solutions on an unbounded domain and thus extend the results of [3, 4]. In [6] similar results were obtained for the case of higher order heat-type equations (see also [6] for more references on the theory of φ -sub-Gaussian processes and partial differential equations with random initial conditions).

The paper is organized as follows. Section 2 collects important definitions and facts on φ -sub-Gaussian processes needed for our study. In Section 3 we consider stochastic processes representing solutions (in $L_2(\Omega)$ sense) of the heat equation with random φ -sub-Gaussian initial conditions and state the results on the distributions of the suprema for such stochastic processes considered over bounded and unbounded domains. Section 4 discusses the conditions of convergence of the integral functional which represent the solution to the random heat equation. This consideration is based on the approach of paper [1] and a result from paper [8] (presented in Appendix).

2 Preliminaries

We present basic definitions and facts from the theory of φ -sub-Gaussian variables and processes which will be used in the paper.

Definition 2.1. [2, 10] Let $\varphi = \{\varphi(x), x \in \mathbb{R}\}$ be a continuous even convex function. The function φ is an Orlicz N-function if $\varphi(0) = 0, \varphi(x) > 0$ as $x \neq 0$ and the following conditions hold: $\lim_{x \rightarrow 0} \frac{\varphi(x)}{x} = 0, \lim_{x \rightarrow \infty} \frac{\varphi(x)}{x} = \infty$.

Condition Q. Let φ be an N-function which satisfies $\liminf_{x \rightarrow 0} \frac{\varphi(x)}{x^2} = c > 0$, where the case $c = \infty$ is possible.

Definition 2.2. [2, 7] Let φ be an N-function satisfying condition Q and $\{\Omega, L, \mathbf{P}\}$ be a standard probability space. The random variable ζ belongs to the space $Sub_\varphi(\Omega)$, if $E\zeta = 0, E \exp\{\lambda\zeta\}$ exists for all $\lambda \in \mathbb{R}$ and there exists a constant $a > 0$ such that the following inequality holds for all $\lambda \in \mathbb{R}$

$$E \exp\{\lambda\zeta\} \leq \exp\{\varphi(\lambda a)\}.$$

The space $Sub_\varphi(\Omega)$ is a Banach space with respect to the norm

$$\tau_\varphi(\zeta) = \sup_{\lambda \neq 0} \frac{\varphi^{(-1)}(\ln E \exp(\lambda\zeta))}{|\lambda|},$$

which can be written equivalently as $\tau_\varphi(\zeta) = \inf\{a > 0 : E \exp\{\lambda\zeta\} \leq \exp\{\varphi(a\lambda)\}$, and it is called the φ -sub-Gaussian standard of the random variable ζ .

Definition 2.3. [5] A family Δ of random variables $\zeta \in Sub_\varphi(\Omega)$ is called strictly φ -sub-Gaussian if there exists a constant C_Δ such that

for all countable sets I of random variables $\zeta_i \in \Delta, i \in I$, the following inequality holds:

$$\tau_\varphi\left(\sum_{i \in I} \lambda_i \zeta_i\right) \leq C_\Delta \left(E \left(\sum_{i \in I} \lambda_i \zeta_i\right)^2\right)^{1/2}. \quad (2.1)$$

The constant C_Δ is called the *determining* constant of the family Δ .

The linear closure of a strictly φ -sub-Gaussian family Δ in the space $L_2(\Omega)$ is the strictly φ -sub-Gaussian with the same determining constant ([5]).

Definition 2.4. [5] Random process $\zeta = \{\zeta(t), t \in T\}$ is called (strictly) φ -sub-Gaussian if the family of random variables $\{\zeta(t), t \in T\}$ is (strictly) φ -sub-Gaussian (with the determining constant C_ζ).

Let K be a deterministic kernel and suppose that the process $X = \{X(t), t \in T\}$ can be represented in the form $X(t) = \int_T K(t, s) d\xi(s)$, where $\xi(t), t \in T$, is a strictly φ -sub-Gaussian random process and the integral above is defined in the mean-square sense. Then the process $X(t), t \in T$, is strictly φ -sub-Gaussian random process with the same determining constant (see [5]).

Definition 2.5. [2, 10] Let $\varphi = \{\varphi(x), x \in \mathbb{R}\}$ be an N-function. The function φ^* defined by

$$\varphi^*(x) = \sup_{y \in \mathbb{R}} (xy - \varphi(y))$$

is called the Young-Fenchel transform (or convex conjugate) of the function φ .

The Young-Fenchel transform is also known as the Legendre (or Legendre-Fenchel) transform, especially in the convex analysis and in the theory of large deviations.

The function φ^* plays an important role in the theory of φ -sub-Gaussian random variables and processes.

If ζ is a φ -sub-Gaussian random variable, then we can write exponential estimates for its tail probabilities, in particular, for all $u > 0$ we have the following bound:

$$P\{|\zeta| > u\} \leq 2 \exp\left\{-\varphi^*\left(\frac{u}{\tau_\varphi(\zeta)}\right)\right\}. \quad (2.2)$$

By using the entropy methods, various estimates for the distribution of suprema of φ -sub-Gaussian random processes can be written in terms of the function φ^* (see [2]).

To derive the main results in Section 3 we will need additional notions and statements.

Lemma 2.1. *Let $Z(u), u \geq 0$ be a continuous monotonically increasing function such that $Z(u) > 0$ and $\frac{u}{Z(u)}$ is nondecreasing for $u \geq u_0$, where $u_0 \geq 0$ is a constant. Then for $u > 0, v > 0$*

$$\min\left(\frac{u}{v}, 1\right) < \frac{Z(u + u_0)}{Z(v + u_0)}.$$

Proof. For $0 < u < v$ we have

$$\frac{u}{v} < \frac{u + u_0}{v + u_0} \leq \frac{Z(u + u_0)}{Z(v + u_0)},$$

since the function $\frac{u}{Z(u)}$ is nondecreasing for $u \geq u_0$. $\square$

Definition 2.6. [6] The function $Z(u), u \geq 0$, is called admissible for the space $Sub_\phi(\Omega)$, if for Z the conditions of Lemma 2.1 hold and for some $\varepsilon > 0$

$$\int_0^\varepsilon \Psi\left(\ln\left(Z^{(-1)}\left(\frac{1}{s}\right) - u_0\right)\right) ds < \infty,$$

where $\Psi(v) = \frac{v}{\phi^{(-1)}(v)}, v > 0$.

Consider a separable φ -sub-Gaussian process defined on a separable metric space (T, d) , where $T = \{a_i \leq t_i \leq b_i, i = 1, 2\}$ and $d(t, s) = \max_{i=1,2} |t_i - s_i|$, $t = (t_1, t_2)$, $s = (s_1, s_2)$.

Theorem 2.1. [6] *Assume that $X = \{X(t), t \in T\}$ is a separable φ -sub-Gaussian process such that*

$$\sup_{\substack{d(t,s) \leq h, \\ t,s \in T}} \tau_\varphi(X(t) - X(s)) \leq \sigma(h), \quad (2.3)$$

where $\{\sigma(h), 0 < h \leq \max_{i=1,2} |b_i - a_i|\}$ is a monotonically increasing continuous function such that $\sigma(h) \rightarrow 0$ as $h \rightarrow 0$ and for some $\varepsilon > 0$

$$\int_0^\varepsilon \Psi\left(\ln \frac{1}{\sigma^{(-1)}(u)}\right) du < \infty, \quad (2.4)$$

where $\Psi(v) = \frac{v}{\varphi^{(-1)}(v)}$. Then

$$P\left\{\sup_{t \in T} |X(t)| > u\right\} \leq 2A(u, \theta)$$

for all $0 < \theta < 1$ and

$$u > \frac{2I_\varphi(\min(\theta\varepsilon_0, \gamma_0))}{\theta(1 - \theta)},$$

where

$$A(u, \theta) = \exp\left\{-\varphi^*\left(\frac{1}{\varepsilon_0}\left(u(1 - \theta) - \frac{2}{\theta}I_\varphi(\min(\theta\varepsilon_0, \gamma_0))\right)\right)\right\},$$

and

$$\varepsilon_0 = \sup_{t \in T} \tau_\varphi(X(t)), \quad \gamma_0 = \sigma\left(\max_{i=1,2} |b_i - a_i|\right),$$

$\varphi^*(u)$ is the Young-Fenchel transform of the function φ ,

$$I_\varphi(\delta) =$$

$$\int_0^\delta \Psi\left[\ln\left[\left(\frac{b_1 - a_1}{2\sigma^{(-1)}(u)} + 1\right)\left(\frac{b_2 - a_2}{2\sigma^{(-1)}(u)} + 1\right)\right]\right] du.$$

3 Main results

Consider the Cauchy problem for the heat equation

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2}, t > 0, x \in \mathbb{R}, \mu > 0, \quad (3.1)$$

subject to the random initial condition

$$u(0, x) = \eta(x), x \in \mathbb{R}, \quad (3.2)$$

where $\eta(x), x \in \mathbb{R}$, is a stochastic process.

In [3, 4] the case of a sub-Gaussian process η was studied. We suppose that η is φ -sub-Gaussian process satisfying the next assumption.

H.1 $\eta(x), x \in \mathbb{R}$ is a real, measurable, mean-square continuous stationary (in wide sense) stochastic process, which is strictly φ -sub-Gaussian with the determining constant c_η .

Let $B(x), x \in \mathbb{R}$, be a covariance function of η , therefore, we have the representation

$$B(x) = \int_{\mathbb{R}} \cos(\lambda x) dF(\lambda), \quad (3.3)$$

where $F(\lambda)$ is a spectral measure, and for η the spectral representation holds

$$\eta(x) = \int_{\mathbb{R}} e^{i\lambda x} Z(d\lambda). \quad (3.4)$$

The stochastic integral is considered as $L_2(\Omega)$ integral. The complex-valued orthogonal random measure Z is such that $E|Z(d\lambda)|^2 = F(d\lambda)$.

Consider the process $u(t, x), t > 0, x \in \mathbb{R}$, defined by

$$u(t, x) = \int_{\mathbb{R}} g(t, x - y) \eta(y) dy, \quad (3.5)$$

where

$$g(t, x) = \frac{1}{(4\pi\mu t)^{1/2}} \exp \left\{ -\frac{x^2}{4\mu t} \right\}, t > 0, x \in \mathbb{R}, \quad (3.6)$$

is the fundamental solution to the heat equation (3.1).

Taking into account (3.4), we can write the following representation of the process given by (3.5):

$$u(t, x) = \int_{\mathbb{R}} \exp \{ i\lambda x - \mu t \lambda^2 \} Z(d\lambda). \quad (3.7)$$

The process (3.7) can be interpreted as the mean-square or $L_2(\Omega)$ solution to the Cauchy problem (3.1)–(3.2), as justified in [4].

Theorem 3.1. *Let $u(t, x), t > 0, x \in \mathbb{R}$, be the stochastic process given by (3.7) and assumption H.1 hold. Assume that $Z(u), u \geq 0$, is a function satisfying conditions of Lemma 2.1 and the following integral converges:*

$$C_Z^2 = \int_{\mathbb{R}} \left(Z^2(\mu\lambda^2 + u_0) + 4Z^2\left(\frac{|\lambda|}{2} + u_0\right) \right) F(d\lambda) < \infty. \quad (3.8)$$

Then

$$\begin{aligned} \sigma(h) &:= \sup_{\max\{|t-t_1|, |x-x_1|\} \leq h} \tau_\varphi(u(t, x) - u(t_1, x_1)) \leq \\ &\leq c_\eta C_Z \left(Z\left(\frac{1}{h} + u_0\right) \right)^{-1}. \end{aligned} \quad (3.9)$$

Proof. The proof is similar to the proof of Theorem 3.1 in [4]. We present the main steps. We have:

$$\begin{aligned} &\sup_{\max\{|t-t_1|, |x-x_1|\} \leq h} \tau_\varphi(u(t, x) - u(t_1, x_1)) \leq \\ &\leq c_\eta \sup_{\max\{|t-t_1|, |x-x_1|\} \leq h} \left(\mathbb{E}(u(t, x) - u(t_1, x_1))^2 \right)^{1/2}. \end{aligned} \quad (3.10)$$

Using calculations from [4], we get

$$\mathbb{E}(u(t, x) - u(t_1, x_1))^2 = \int_{\mathbb{R}} |b(\lambda)|^2 F(d\lambda), \quad (3.11)$$

where

$$\begin{aligned} b(\lambda) &= \exp\{i\lambda x\} \exp\{-\mu\lambda^2 t\} \\ &\quad - \exp\{i\lambda x_1\} \exp\{-\mu\lambda^2 t_1\}. \end{aligned}$$

From formula (3.11) from [4] we have for $|t - t_1| \leq h$ and $|x - x_1| \leq h$:

$$|b(\lambda)| \leq \left(1 - \exp \left\{ -\mu\lambda^2 |t - t_1| \right\} \right)^2 +$$

$$\begin{aligned} &+ 4 \sin^2 \left(\frac{1}{2} \lambda (x - x_1) \right) \leq \left(\min(\mu\lambda^2 h, 1) \right)^2 + \\ &+ 4 \left(\min \left(\frac{h}{2} |\lambda|, 1 \right) \right)^2. \end{aligned}$$

Using Lemma 2.1 we can write the bound

$$|b(\lambda)| \leq \frac{Z(\mu\lambda^2 + u_0)}{Z(\frac{1}{h} + u_0)} + 4 \frac{Z(\frac{|\lambda|}{2} + u_0)}{Z(\frac{1}{h} + u_0)}. \quad (3.12)$$

Substituting (3.12) in (3.11) and using inequality (3.10), we obtain (3.9). $\square$

Theorem 3.2. *Let $u(t, x), a \leq t \leq b, c \leq x \leq d$, be a separable modification of the stochastic process given by (3.7) and assumption H.1 hold. Assume that $Z(u), u \geq 0$, is an admissible function for the space $\text{Sub}_\phi(\Omega)$, and the integral (3.8) converges. Then for $0 < \theta < 1$ and*

$$u > \frac{2\hat{I}_\varphi(\min(\theta\Gamma, \gamma_0))}{\theta(1 - \theta)}$$

the following inequality holds true:

$$\mathbb{P} \left\{ \sup_{\substack{a \leq t \leq b; \\ c \leq x \leq d}} |u(t, x)| > u \right\} \leq 2A(u, \theta), \quad (3.13)$$

where

$A(u, \theta)$

$$\begin{aligned} &= \exp \left\{ -\varphi^* \left(\frac{1}{\Gamma} (u(1 - \theta) - \frac{2}{\theta} \hat{I}_\varphi(\min(\theta\Gamma, \gamma_0))) \right) \right\}; \\ &\hat{I}_\varphi(\sigma) \\ &= \int_0^\sigma \Psi \left(\ln \left[\left(\frac{b-a}{2} \left(Z^{(-1)} \left(\frac{c_\eta C_Z}{s} \right) - u_0 \right) + 1 \right) \times \right. \right. \\ &\quad \left. \left. \times \left(\frac{d-c}{2} \left(Z^{(-1)} \left(\frac{c_\eta C_Z}{s} \right) - u_0 \right) + 1 \right) \right] \right) ds; \end{aligned} \quad (3.14)$$

$$\Gamma = c_\eta \left(\int_{\mathbb{R}} F(d\lambda) \right)^{1/2}, \quad \Psi(u) = \frac{u}{\varphi^{(-1)}(u)},$$

$$\gamma_0 = \frac{c_\eta C_Z}{Z(\frac{1}{\varkappa} + u_0)}, \quad \varkappa = \max(b - a, d - c);$$

C_Z is defined in (3.8).

Proof. The assertion of this theorem follows from Theorems 2.1 and 3.1.

Note, that from Theorem 3.1 we have that condition (2.3) of Theorem 2.1 holds with $\sigma(h) = c_\eta C_Z \left(Z\left(\frac{1}{h} + u_0\right) \right)^{-1}$. We also have

$$\begin{aligned} \varepsilon_0 &= \sup_{(t,x) \in [a,b] \times [c,d]} \tau_\phi(u(t,x)) \leq \\ &\leq c_\eta \sup_{(t,x) \in [a,b] \times [c,d]} \left(\mathbb{E}(u(t,x))^2 \right)^{1/2} \leq \\ &\leq c_\eta \sup_{(t,x) \in [a,b] \times [c,d]} \left(\int_{\mathbb{R}} \exp\{-2\mu\lambda^2 t\} F(d\lambda) \right)^{1/2} \leq \\ &\leq c_\eta \left(\int_{\mathbb{R}} F(d\lambda) \right)^{1/2} =: \Gamma < \infty. \end{aligned}$$

□

Example 3.1. Let $\eta = \{\eta(u), u \in \mathbb{R}\}$ be a centered Gaussian random process satisfying Assumption H.1. Then $c_\eta = 1$, $\varphi(x) = \frac{x^2}{2}$, $\varphi^*(x) = \frac{x^2}{2}$, $\Psi(x) = \frac{1}{\sqrt{2}}x^{1/2}$. Consider the following admissible function

$$Z(u) = \ln^\alpha(u+1), \quad u \geq 0, \quad \alpha > 1/2.$$

In this case

$$u_0 = e^\alpha - 1, \quad Z^{(-1)}(v) = \exp\left\{v^{\frac{1}{\alpha}}\right\} - 1,$$

$$Z(v + u_0) = \ln^\alpha(v + e^\alpha),$$

$$C_Z^2 = \int_{\mathbb{R}} \left[\ln^{2\alpha}(\mu\lambda^2 + e^\alpha) + 4 \ln^{2\alpha}\left(\frac{|\lambda|}{2} + e^\alpha\right) \right] F(d\lambda).$$

The above integral converges if the following integral converges

$$\int_{\mathbb{R}} \ln^{2\alpha}(|\lambda| + e^\alpha) F(d\lambda). \quad (3.15)$$

That is, if condition (3.15) holds true, then Theorem 3.2 holds. It follows from (3.14) that

$$\begin{aligned} \hat{I}_\varphi(\delta) &= \int_0^\delta \frac{1}{\sqrt{2}} \left(\ln \left[\left(\frac{b-a}{2} \left(\exp\left\{\left(\frac{C_Z}{s}\right)^\alpha\right\} - e^\alpha \right) + 1 \right) \times \left(\frac{d-c}{2} \left(\exp\left\{\left(\frac{C_Z}{s}\right)^\alpha\right\} - e^\alpha \right) + 1 \right) \right] \right)^{\frac{1}{2}} ds. \end{aligned}$$

Let $\frac{d-c}{2} e^\alpha > 1$ and $\frac{b-a}{2} e^\alpha > 1$, then

$$\begin{aligned} \hat{I}_\varphi(\delta) &\leq \frac{1}{\sqrt{2}} \\ &\times \int_0^\delta \left(\ln \left(\frac{d-c}{2} \frac{b-a}{2} \exp\left\{2\left(\frac{C_Z}{s}\right)^\alpha\right\} \right) \right)^{\frac{1}{2}} ds \\ &\leq \frac{1}{\sqrt{2}} \int_0^\delta \ln \left(\frac{(d-c)(b-a)}{4} \right)^{\frac{1}{2}} ds \\ &\quad + \int_0^\delta \left(\frac{C_Z}{s} \right)^{\frac{1}{2\alpha}} ds \\ &= \frac{\delta}{\sqrt{2}} \left(\ln \left(\frac{(d-c)(b-a)}{4} \right) \right)^{\frac{1}{2}} \\ &\quad + \delta \left(\frac{C_Z}{\delta} \right)^{\frac{1}{2\alpha}} \left(1 - \frac{1}{2\alpha} \right)^{-1}. \end{aligned} \quad (3.16)$$

It follows from (3.13) that in this case for

$$u > \frac{2\hat{I}_\varphi(\min(\theta\Gamma, \gamma_0))}{\theta(1-\theta)}$$

we have

$$\begin{aligned} P\left\{ \sup_{\substack{a \leq t \leq b, \\ c \leq x \leq d}} |u(t,x)| > u \right\} &\leq \\ &\leq \exp \left\{ -\frac{1}{2} \left(\frac{1}{\Gamma} \left(u(1-\theta) - \frac{2}{\theta} \hat{I}_\varphi(\min(\theta\Gamma, \gamma_0)) \right) \right)^2 \right\}, \\ \gamma_0 &= \frac{C_Z}{\ln^\alpha\left(\frac{1}{\varepsilon} + e^\alpha\right)}. \end{aligned}$$

If θ is such that $\theta\Gamma < \gamma_0$ ($\theta < \frac{\gamma_0}{\Gamma}$), then for

$$u > \sup_{0 < \theta < \frac{\gamma_0}{\Gamma}} \frac{2\hat{I}_\varphi(\theta\Gamma)}{\theta(1-\theta)}$$

$$\begin{aligned} P\left\{ \sup_{\substack{a \leq t \leq b, \\ c \leq x \leq d}} |u(t,x)| > u \right\} &\leq \\ &\leq \inf_{0 < \theta < \frac{\gamma_0}{\Gamma}} \exp \left\{ -\frac{1}{2} \left(\frac{1}{\Gamma} \left(u(1-\theta) - \frac{2}{\theta} \hat{I}_\varphi(\theta\Gamma) \right) \right)^2 \right\}, \end{aligned}$$

and if $\frac{\gamma_0}{\Gamma} > 1$ then for

$$u > \sup_{0 < \theta < 1} \frac{2\hat{I}_\varphi(\theta\Gamma)}{\theta(1-\theta)}$$

we get

$$\begin{aligned} P\left\{ \sup_{\substack{a \leq t \leq b, \\ c \leq x \leq d}} |u(t,x)| > u \right\} &\leq \\ &\leq \inf_{0 < \theta \leq 1} \exp \left\{ -\frac{1}{2} \left(\frac{1}{\Gamma} \left(u(1-\theta) - \frac{2}{\theta} \hat{I}_\varphi(\theta\Gamma) \right) \right)^2 \right\}. \end{aligned}$$

Example 3.2. Let $\eta = \{\eta(u), u \in \mathbb{R}\}$ be a centered Gaussian random process, as in Example 3.1. Consider the admissible function $Z(u) = |u|^\alpha$, $0 < \alpha \leq 1$. In this case

$$u_0 = 0, \quad Z^{(-1)}(u) = u^{\frac{1}{\alpha}}, \quad u > 0,$$

$$C_Z^2 = \int_{\mathbb{R}} \left((\mu\lambda^2)^{2\alpha} + 4\left(\frac{|\lambda|}{2}\right)^{2\alpha} \right) F(d\lambda)$$

$$\leq \int_{\mathbb{R}} \left((\mu\lambda^2)^{2\alpha} + 4^{1-\alpha}\lambda^{2\alpha} \right) F(d\lambda). \quad (3.17)$$

This integral converges if the next integral converges

$$\int_{\mathbb{R}} \lambda^{4\alpha} F(d\lambda) < \infty. \quad (3.18)$$

That is, if condition (3.18) holds, then Theorem 3.2 holds. It follows from (3.14) that

$$\hat{I}_\varphi(\delta) = \frac{1}{\sqrt{2}} \int_0^\delta \left(\ln \left[\left(\frac{b-a}{2} \left(\frac{C_Z}{s} \right)^{\frac{1}{\alpha}} + 1 \right) \times \left(\frac{d-c}{2} \left(\frac{C_Z}{s} \right)^{\frac{1}{\alpha}} + 1 \right) \right] \right)^{\frac{1}{2}} ds.$$

For $0 \leq \beta < 1$, $x > 0$, $y > 0$, we have

$$\begin{aligned} \ln((1+x)(1+y)) &= \frac{1}{\beta} \ln[(1+x)(1+y)]^\beta \\ &= \frac{1}{\beta} [\ln(1+x)^\beta + \ln(1+y)^\beta] \\ &\leq \frac{1}{\beta} [\ln(1+x^\beta) + \ln(1+y^\beta)] \\ &\leq \frac{1}{\beta} (x^\beta + y^\beta), \end{aligned}$$

therefore, in the case of $\beta < \alpha$ we obtain

$$\begin{aligned} \hat{I}_\varphi(\delta) &\leq \frac{1}{\sqrt{2}\beta} (C_Z)^{\frac{\beta}{2\alpha}} \delta^{(1-\frac{\beta}{2\alpha})} \frac{1}{1-\frac{\beta}{2\alpha}} \left[\left(\frac{b-a}{2} \right)^{\frac{\beta}{2}} \right. \\ &\quad \left. + \left(\frac{d-c}{2} \right)^{\frac{\beta}{2}} \right] =: \hat{J}_\varphi(\delta, \beta). \end{aligned} \quad (3.19)$$

It follows from (3.13) that in this case for

$$u > \frac{2\hat{J}_\varphi(\min(\theta\Gamma, \gamma_0), \beta)}{\theta(1-\theta)}, \quad \gamma_0 = C_Z \varkappa^\alpha$$

we get the estimate

$$\begin{aligned} P\left\{ \sup_{\substack{a \leq t \leq b \\ c \leq x \leq d}} |u(t, x)| > u \right\} &\leq \\ &\leq \inf_{\theta, \beta} \exp \left\{ -\frac{1}{2} \left(\frac{1}{\Gamma} \left(u(1-\theta) \right. \right. \right. \\ &\quad \left. \left. \left. - \frac{2}{\theta} \hat{J}_\varphi(\min(\theta\Gamma, \gamma_0), \beta) \right) \right)^2 \right\}. \end{aligned}$$

Consider now the process $u(t, x)$, $(t, x) \in V$ defined on an unbounded domain of the following form: $V = [-A, A] \times [0, \infty)$, $A > 0$.

Introduce a family of segments $\{[b_k, b_{k+1}], k = 0, 1, \dots\}$ such that $b_0 = 0, b_k > b_{k+1}, b_k \rightarrow \infty, k \rightarrow \infty; b_{k+1} - b_k \geq 2A$, define the sets $V_k = [-A, A] \times [b_k, b_{k+1}], k = 0, 1, \dots$ and $V = \bigcup_{k=0}^\infty V_k$.

Let $\{a(t), t \geq 0\}$ be a continuous strictly increasing function such that $a(t) > 0, t \geq 0$ and $a(t) \rightarrow \infty$, as $t \rightarrow \infty$.

Denote:

$$a_k = \min_{t \in [b_k, b_{k+1}]} a(t); \quad \varepsilon_k = \sup_{(x, t) \in V_k} \tau_\varphi(u(t, x)).$$

In the next theorem the rate of growth of $u(t, x)$, $(t, x) \in V$, is evaluated.

Theorem 3.3. Let $u(t, x)$, $(t, x) \in V$, be a separable modification of the stochastic process given by (3.7) and assumption H.1 hold. Assume that $Z(u), u \geq 0$, is an admissible function for the space $\text{Sub}_\varphi(\Omega)$, and the integral (3.8) converges.

Suppose that the following conditions hold:

1. $\sup_{k=0, \infty} \frac{\varepsilon_k}{a_k} < \infty$.
2. For $\sigma > 0$

$$\begin{aligned} \hat{I}_\varphi(\sigma) &= \int_0^\sigma \Psi \left(\ln \left[\left(A(Z^{(-1)}(\frac{c_\eta C_Z}{s}) - u_0 \right) + 1 \right) \times \left(\frac{b_{k+1} - b_k}{2} (Z^{(-1)}(\frac{c_\eta C_Z}{s}) - u_0) + 1 \right) \right] \right) ds \end{aligned}$$

and for some $\theta, 0 < \theta < 1$

$$\sup_{k=0, \infty} \frac{\hat{I}_{\varphi, k}(\theta \varepsilon_k)}{a_k} < \infty.$$

3. The series $\sum_{k=0}^\infty \exp \left\{ -\varphi^* \left(\frac{sa_k(1-\theta)}{2\varepsilon_k} \right) \right\}$ converges for some s such that $\sup_{k=0, \infty} \frac{4\varepsilon_k}{a_k(1-\theta)} < s < \frac{u}{2}$.

Then for

$$\begin{aligned} u &> \sup_{k=0, \infty} \frac{\hat{I}_{\varphi, k}(\theta \varepsilon_k)}{a_k} \frac{4}{\theta(1-\theta)}, \\ P\left\{ \sup_{(t, x) \in V} \frac{|u(t, x)|}{a(t)} > u \right\} &\leq 2 \exp \left\{ -\varphi^* \left(\frac{u}{s} \right) \right\} \\ &\times \sum_{k=0}^\infty \exp \left\{ -\varphi^* \left(\frac{sa_k(1-\theta)}{2\varepsilon_k} \right) \right\} =: 2A(u). \end{aligned} \quad (3.20)$$

Proof. The assertion of this theorem follows from Theorem 6.1 ([6]) (see also Theorem 3 in [9]). The arguments used for the proof are the same as that for Theorem 6.2. in [6]. $\square$

Corollary 3.1. *Let the assumptions of Theorem 3.3 hold true. Then there exists a random variable ξ such that*

$$P\{\xi > u\} \leq 2A(u)$$

for

$$u > \sup_{k=0,\infty} \frac{\hat{I}_{\varphi,k}(\theta\varepsilon_k)}{a_k} \frac{4}{\theta(1-\theta)}$$

and

$$|u(x, t)| < \xi a(t)$$

with probability one.

4 On convergence of the integral (3.7)

In the papers [1], [6] the integral functionals (with kernels of a particular form) of φ -sub-Gaussian random processes were studied as solutions of higher order partial differential equations with random initial conditions.

By using the same approach as in [1], we can state that under the conditions of Theorem 3.2 the integrals given by formula (3.7), that is,

$$u(t, x) = \int_{\mathbb{R}} \exp \left\{ i\lambda x - \mu t \lambda^2 \right\} Z(d\lambda)$$

converge with probability 1 for $|x| \leq A$, $0 \leq t \leq T$, where $A > 0$ and $T > 0$ are some constants. For this end we can apply the result from [8] (see Theorem A.1 in Appendix). In view of Theorem A.1 it is sufficient to show that there exists a sequence $a_n > 0$, $a_n \rightarrow \infty$ as $n \rightarrow \infty$, such that the sequence $u_n(t, x) = \int_{-a_n}^{a_n} \exp \left\{ i\lambda x - \mu t \lambda^2 \right\} Z(d\lambda)$ converges uniformly in probability for $|x| \leq A$, $0 \leq t \leq T$.

Reconsidering the proof of Theorem 3.1, we obtain

$$\begin{aligned} & \sup_{\max\{|t-t_1|, |x-x_1|\} \leq h} \tau_{\varphi}(u_n(t, x) - u_n(t_1, x_1)) \\ & \leq c_{\eta} \sup_{\max\{|t-t_1|, |x-x_1|\} \leq h} \left(\mathbb{E}(u_n(t, x) - u_n(t_1, x_1))^2 \right)^{1/2} \\ & \leq c_{\eta} \int_{\mathbb{R}} |b(\lambda)|^2 F(d\lambda) \leq c_{\eta} C_Z \left(Z \left(\frac{1}{h} + u_0 \right) \right)^{-1}, \end{aligned}$$

therefore, condition (5.21) of Theorem A.1 holds with

$$\sigma(h) = c_{\eta} C_Z \left(Z \left(\frac{1}{h} + u_0 \right) \right)^{-1}.$$

Taking into account that

$$\sigma^{(-1)}(\varepsilon) = \frac{1}{Z^{(-1)} \left(\frac{c_{\eta} C_Z}{\varepsilon} \right) - u_0}, \quad 0 < \varepsilon < \frac{Z(u_0)}{c_{\eta} C_Z},$$

and Z is an admissible function (see Definition 2.6), we conclude that condition (5.22) of Theorem A.1 holds as well.

5 Appendix

Let (T, d) be a compact metric space and $C(T)$ be the Banach space of continuous functions with uniform norm. Let $X_k = \{X_k(t), t \in T\}$ be a sequence of φ -subGaussian random processes such that $X_k \in C(T)$. The general conditions of convergence in probability of X_k in the space $C(T)$ are presented in the book [2]. In the paper [8] these conditions are presented for the case where T is a finite-dimensional space.

Theorem A.1. ([8]) *Let R^k be a k -dimensional space, $d(t, s) = \max_{1 \leq i \leq k} |t_i - s_i|$, $T = \{0 \leq t_i \leq T_i, i = 1, 2, \dots, k\}$, $T_i > 0$; $X_n = \{X_n(t), t \in T\}$ be a sequence of φ -subGaussian random processes such that $X_n \in C(T)$. Let us assume also that there exists a continuous increasing function $\sigma = \{\sigma(h), h > 0\}$, $\sigma(h) \rightarrow 0$ as $h \rightarrow 0$, such that*

$$\sup_{d(t,s) \leq h} \tau_{\varphi}(X_n(t) - X_n(s)) \leq \sigma(h) \quad (5.21)$$

and

$$\int_{0+} \Psi \left(\ln \frac{1}{\sigma^{(-1)}(\varepsilon)} \right) d\varepsilon < \infty, \quad (5.22)$$

where $\Psi(u) = \frac{u}{\varphi^{(-1)}(u)}$, $\sigma^{(-1)}(u)$ is the inverse function of $\sigma(u)$, $\varphi^{(-1)}(u)$ is the inverse function of $\varphi(u)$, for $u > 0$, and $\int_{0+} f(\varepsilon) d\varepsilon$ denotes $\int_0^{\delta} f(\varepsilon) d\varepsilon$ for sufficiently small $\delta > 0$. If the sequence of processes $X_n(t)$, $n \geq 1$, converges in probability to $X(t)$ for all $t \in T$, then $X_n(t)$ converges in probability to $X(t)$ in the space $C(T)$.

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Received: 15.01.2020