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В. Голомозий¹, кандидат наук, доцент
С. Шаріпов², аспірант

Про центральні граничні теореми для гіллястих процесів із залежною міграцією

¹Київський національний університет імені Тараса Шевченка, 01033, Київ, вул. Володимирська 64.

²Інститут Математики імені В.І. Романовського АН РУз, вул. Мирзо-Улугбека 81, 100170 Ташкент, Узбекистан.

V. Golomoziy¹, Ph.D., Associate Professor
S. Sharipov², Post-graduate

On central limit theorems for branching processes with dependent immigration

¹Taras Shevchenko National University of Kyiv, 01033, Kyiv, 64 Volodymyrska st.

²V.I.Romanovskiy Institute of Mathematics, Academy of Sciences of Uzbekistan, Mirzo Ulughbek str.81, 100170 Tashkent, Uzbekistan.

У даній роботі ми розглядаємо субкритичні та суперкритичні гіллясті процеси з імміграцією, що залежить від покоління частинок. У роботі доведено центральні граничні теореми для флуктуацій гіллястого процесу з m -залежною імміграцією у випадку коли середні значення іммігруючих частинок прямують до нескінченності. Встановлено слабку збіжність флуктуацій суперкритичного процесу до стандартного нормального розподілу за припущення регулярної зміни на нескінченності середнього та дисперсії імміграційного процесу, та слабку збіжність флуктуацій субкритичного процесу до стандартного нормального розподілу без такого припущення.

Ключові слова: Гіллясті процеси, імміграція, m -залежність.

In this paper we consider subcritical and supercritical discrete time branching processes with generation dependent immigration. We prove central limit theorems for fluctuation of branching processes with immigration when the mean of immigrating individuals tends to infinity with the generation number and immigration process is m -dependent. The first result states on weak convergence of the fluctuation subcritical branching processes with m -dependent immigration to standard normal distribution. In this case, we do not assume that the mean and variance of immigration process are regularly varying at infinity. In contrast, in Theorem 3.2, we suppose that the mean and variance are to be regularly varying at infinity. The proofs are based on direct analytic method of probability theory.

Key Words: Branching process, immigration, m -dependence

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1 Introduction

Let $\{Y_{k,i}, k \geq 1, i \geq 1\}$ and $\{\varepsilon_k, k \geq 1\}$ be two sequence of non-negative integer-valued random variables such that the two families $\{Y_{k,i}, k \geq 1, i \geq 1\}$ and $\{\varepsilon_k, k \geq 1\}$ are independent, $\{Y_{k,i}, k \geq 1, i \geq 1\}$ are independent and identically distributed (i.i.d.). We consider branching processes with immigration $\{W(k), k \geq 0\}$ defined recursively as

$$W(0) = 0, \quad W(k) = \sum_{i=1}^{W(k-1)} Y_{k,i} + \varepsilon_k, \quad k \geq 1. \quad (1)$$

We can interpret $Y_{k,i}$ as the number of offsprings produced by the i -th individual belonging to

the $(k-1)$ -th generation and ε_k is the number of immigrants in the k -th generation. In this interpretation $W(k)$ is the number of individuals in the k -th generation. Assume that $a = \mathbb{E}Y_{1,1} < \infty$. Process $W(k)$ is called subcritical, critical or supercritical depending on $a < 1$, $a = 1$ or $a > 1$, respectively.

The asymptotic behavior the distribution of $W(n)$ as $n \rightarrow \infty$ has been studied by many authors, see, e.g., the survey of Vatutin and Zubkov [45]. Early contribution is due to Sevast'yanov [42], who derived limit theorems for continuous-time Markov branching processes when immigration $\{\varepsilon_n, n \geq 1\}$ is an independent and distributed by Poisson law. Then many

research works appeared in which various generalizations of the immigration process were considered. For instance, when the sequence $\{\varepsilon_n, n \geq 1\}$ is i.i.d. the asymptotic behavior of $W(n)$ is well studied (see [10], [40], [41], [12], [30] and references therein). In the critical case, Nagaev [29] considered a wide-sense stationary immigration process $\{\varepsilon_n, n \geq 1\}$, and proved weak convergence of distribution of $n^{-1}W(n)$ as $n \rightarrow \infty$ to a gamma distribution. Asadullin and Nagaev [1] managed to weaken conditions of [29] up to the condition "type of ergodicity". More precisely, they studied weak convergence of $n^{-1}W(n)$ under more general situation that there exists a random variable ε , such that $n^{-1}\mathbb{E}\left|\sum_{k=1}^n (\varepsilon_k - \varepsilon)\right| \rightarrow 0$ as $n \rightarrow \infty$. Badalbaev and Zubkov [5] for the sequence of special random processes (including branching processes with immigration) proved a limit theorem which contains results of [29] and [1] as a special case. Concerning functional limit theorems for (1), we refer to Wei and Winnicki [46], Sriram [43], Li [25], [24], Ispány et al. [14], [15], Khusanbaev [19], [21], Iksanov and Kabluchko [13] and see references therein. We refer to papers [18], [20], [39] where the rates of convergence in central limit theorem for (1) were studied.

While most of previous literature was concerned with stationary immigration, the non-stationary (time-dependent or state-dependent) immigration case were studied by Foster and Williamson [7] and by Pakes [32]. In this class of processes, unlike in classical models, immigration rate may depend on time of immigration. For further investigations for branching processes with non-stationary immigration we refer to Pakes [31], Hering [11], Badalbaev and Rahimov [4], Mitov and Yanev [26]-[28] (see also [2], [3]). Rahimov [33]-[34] studied the case when immigration process is non-stationary and increasing and obtained for all cases central limit theorems (CLT's) for fluctuation of (1). A key reference is a monograph by Rahimov [35], where one can find a variety of results and references related to this process. Later, Rahimov [36] proved functional limit theorems for a fluctuation of critical process defined by (1). For deterministic approximation for $W(n)$, we refer to Rahimov [37], Khusanbaev [16]-[17] and see references therein.

However, in all of previous works, the immigration process was assumed to be independent.

Guo and Zhang [9] proved a functional limit theorem for fluctuations of the critical branching process (1) under the condition of m -dependent immigration. Recently, Khusanbaev et al. [22]-[23] obtained functional limit theorems for fluctuation of (1) when immigration process satisfies ϕ -mixing condition. The results of above cited papers showed that the limit behavior of fluctuations of process (1) clearly depends on the rate of convergence to infinity of the mean number of immigrants.

The purpose of this paper is to establish CLTs for fluctuations of subcritical and supercritical process $W(n)$ defined by (1) in the case when the immigration process is m -dependent and the mean of immigrants tends to infinity.

The paper is organized as follows. In Section 2, we provide basic facts and definitions. Section 3 contains main results and their proofs.

2 Notations and definitions

We begin by introducing basic facts. First, let us recall the notation of m -dependence.

Означення 2.1. It is said that a sequence of random variables $\{X_n, n \geq 1\}$ satisfies the m -dependence condition for some $m \geq 0$ if the random vectors $(X_1, \dots, X_k)$ and $(X_{k+m+1}, \dots)$ are independent for all $k \geq 1$ (see [6]).

Throughout the paper, we will assume that the immigration process is heterogeneous, i.e., random variables $\{\varepsilon_k, k \geq 1\}$ are non-identically distributed and m -dependent, and the particles entering the population act independently of the other particles and according to the same law as particles of the population. We denote by $Z_i^j(k)$, $k = i, i+1, \dots, 1 \leq j \leq \varepsilon_i$, the branching Galton-Watson process generated by j -th particles arriving at the moment i with $Z_i^j(i) = 1$. According to our assumptions branching processes $Z_i^j(k)$, $k \geq i$; $i, j \geq 1$ are independent and $Z_i^j(k+i)$, $k = 0, 1, \dots$ has the same distribution as $Z_1^1(k)$, $k = 1, 2, \dots$.

From now on, we assume that

$$b^2 := \text{Var}(Y_{1,1}) < \infty,$$

$$\gamma := \mathbb{E}Y_{1,1}(Y_{1,1} - 1)(Y_{1,1} - 2) < \infty.$$

We also assume that $\alpha(n) = \mathbb{E}\varepsilon_n < \infty$ and $\beta(n) = \text{Var}(\varepsilon_n) < \infty$ for all $n \in \mathbb{N}$.

Означення 2.2 ([8]). A measurable function $l : (0, \infty) \rightarrow (0, \infty)$ is slowly varying if for all $\lambda > 0$

$$\lim_{x \rightarrow \infty} \frac{l(\lambda x)}{l(x)} = 1.$$

Означення 2.3 ([8]). A measurable function $f : (0, \infty) \rightarrow (0, \infty)$ is called regularly varying at infinity if it can be represented in the form

$$f(x) = x^\rho l(x),$$

where ρ is called index of regular variation and $\rho \in (-\infty, \infty)$ and $l_\rho(x)$ is a slowly varying function.

Означення 2.4 ([8]). A sequence of positive numbers $\{x(n), n \geq 1\}$ is called a regularly varying sequence of index ρ if there is a sequence of positive terms $\{y(n), n \geq 1\}$ satisfying as $n \rightarrow \infty$

$$x(n) \sim Cy(n), \quad n(1 - (y(n-1)/y(n))) \rightarrow \rho < \infty.$$

If a sequence $\{x(n), n \geq 1\}$ is regularly varying with index ρ , we will write $\{x(n), n \geq 1\} \in R_\rho$.

Under the above assumptions $A(a, n) = \mathbb{E}W(n)$ and $B^2(a, n) = \text{Var}(W(n))$ are finite for all $n \geq 1$. Denote

$$f_k(t) = \mathbb{E}e^{itZ_1^{(k)}}, \quad \Psi_k(t) = \mathbb{E}e^{itW^{(k)}}, \quad k \geq 1.$$

Denote by $\Phi(x)$ distribution function of the standard normal law. In this paper, $a_n \sim b_n$ denotes $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1$ and $a_n = o(b_n)$ denotes

$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$, $a \wedge b = \min(a, b)$ and $\xrightarrow{P}$ denotes the convergence of random variables in probability. C denotes a generic constant which may be different from place to place.

3 Main results and their proofs

Now we are ready to formulate our main results. We begin with a simple Lemma which provides formulas for $A(a, n)$ and $B^2(a, n)$ when immigration processes are dependent.

Lemma 1. Let $W(n), n \geq 0$ be a branching processes with immigration given in (1). Then, for all $n \geq 1$,

$$A(a, n) = \sum_{i=1}^n a^{n-i} \alpha(i),$$

$$B^2(a, n) = \Delta^2(a, n) + \sigma^2(a, n) + 2\omega(a, n), \quad (2)$$

where

$$\Delta^2(a, n) = \frac{b^2}{1-a} \sum_{i=1}^n a^{n-i-1} (1-a^{n-i}) \alpha(i), \quad a \neq 1,$$

$$\sigma^2(a, n) = \sum_{i=1}^n a^{2(n-i)} \beta(i),$$

$$\omega(a, n) = \sum_{i=2}^n \sum_{k=1}^{i-1} a^{2n-i-k} \text{cov}(\varepsilon_i, \varepsilon_k).$$

Доведення. By (1) and recurrence, for all $n \geq 1$,

$$A(a, n) = \mathbb{E} \left[\sum_{i=1}^{W(n-1)} Y_{n,i} \right] + \mathbb{E}\varepsilon_n =$$

$$= a\mathbb{E}W(n-1) + \alpha(n) = \dots = \sum_{i=1}^n a^{n-i} \alpha(i),$$

$$B^2(a, n) = \mathbb{E}W(n-1)\text{Var}(Y_{2,1}) +$$

$$+ \text{Var}(W(n-1))(\mathbb{E}(Y_{2,1}))^2 + 2 \text{cov} \left(\sum_{i=1}^{W(n-1)} Y_{n,i}, \varepsilon_n \right)$$

$$= b^2 A(n-1) + a^2 \text{Var}(W(n-1)) +$$

$$+ \beta(n) + 2 \text{cov} \left(\sum_{i=1}^{W(n-1)} Y_{n,i}, \varepsilon_n \right).$$

Note that

$$\text{cov} \left(\sum_{i=1}^{W(n-1)} Y_{n,i}, \varepsilon_n \right) =$$

$$= \mathbb{E} \left[\left(\sum_{i=1}^{W(n-1)} Y_{n,i} - \mathbb{E} \sum_{i=1}^{W(n-1)} Y_{n,i} \right) (\varepsilon_n - \mathbb{E}\varepsilon_n) \right]$$

$$= \mathbb{E}[(aW(n-1) - a\mathbb{E}W(n-1))(\varepsilon_n - \mathbb{E}\varepsilon_n)] =$$

$$= a \text{cov}(W(n-1), \varepsilon_n) =$$

$$= a\mathbb{E} \left[\left(\sum_{i=1}^{W(n-2)} Y_{n-1,i} + \varepsilon_{n-1} - a\mathbb{E}W(n-2) - \mathbb{E}\varepsilon_{n-1} \right) (\varepsilon_n - \mathbb{E}\varepsilon_n) \right]$$

Therefore

$$= a \mathbb{E} \left[\left(\sum_{i=1}^{W(n-2)} Y_{n-1,i} - a \mathbb{E} W(n-2) + \varepsilon_{n-1} - \mathbb{E} \varepsilon_{n-1} \right) \left(\varepsilon_n - \mathbb{E} \varepsilon_n \right) \right] = e^{-it \frac{A(a,n)}{B(a,n)}} \mathbb{E} \prod_{k=1}^n f_{n-k}^{\varepsilon_k} \left(\frac{t}{B(a,n)} \right).$$

$$= a^2 \text{cov}(W(n-2), \varepsilon_n) + a \text{cov}(\varepsilon_{n-1}, \varepsilon_n) =$$

$$\dots = \sum_{j=1}^{n-1} a^{n-j} \text{cov}(\varepsilon_j, \varepsilon_n).$$

By iteration, we get (2). The Lemma 1 is proved. $\square$

First result deals with CLT for a fluctuation of subcritical branching process (1) with m -dependent and increasing immigration.

Theorem 3.1. *Let $a < 1$, $\gamma < \infty$ and the immigration process $\{\varepsilon_n, n \geq 1\}$ is m -dependent. Suppose $\alpha(n) \in R_\alpha$ and $\beta(n) \in R_\beta$ with $\alpha, \beta \geq 0$. If $\alpha(n) \rightarrow \infty$ and $\beta(n) = o(\alpha(n))$ as $n \rightarrow \infty$, then, as $n \rightarrow \infty$*

$$\mathbb{P} \left(\frac{W(n) - A(a,n)}{B(a,n)} < x \right) \rightarrow \Phi(x), \quad x \in \mathbb{R}.$$

Доведення. First of all, we note that the process $W(k)$ can be represented as

$$W(k) = \sum_{i=1}^k \sum_{j=1}^{\varepsilon_i} Z_i^j(k), \quad k \geq 1. \quad (3)$$

By (3), in view of the independence of random variables $Z_i^j(k)$, it follows

$$\Psi_n(t) = \mathbb{E} \prod_{k=1}^n f_{n-k}^{\varepsilon_k}(t).$$

Indeed, $Z_i^j(k)$ are independent of $\varepsilon_n, n \geq 1$, since by assumption two sequences $\{Y_{k,i}, k \geq 1, i \geq 1\}$ and $\{\varepsilon_n, n \geq 1\}$ are independent, obtain

$$\Psi_n(t) = \mathbb{E} \exp(itW(n)) = \mathbb{E} \exp(it \sum_{i=1}^n \sum_{j=1}^{\varepsilon_i} Z_i^j(n))$$

$$\begin{aligned} &= \mathbb{E} [\mathbb{E} [\prod_{i=1}^n \exp(it \sum_{j=1}^{\varepsilon_i} Z_i^j(n)) | \sigma(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)]] = \\ &= \mathbb{E} [\prod_{i=1}^n [\mathbb{E} (it Z_1^1(n))]^{\varepsilon_i}]. \end{aligned}$$

Since $\mathbb{E}[Z_1^1(k)]^3 < \infty$ then the Taylor series expansion is valid for the characteristic function $f_k(t)$

$$f_k(t) = 1 + it \mathbb{E} Z_1^1(k) -$$

$$- \frac{t^2}{2} \mathbb{E} [Z_1^1(k)]^2 + \theta_k \frac{it^3}{6} \mathbb{E} [Z_1^1(k)]^3,$$

where $|\theta_k| \leq 1, k \geq 1$.

It is well-known (see [3])

$$\mathbb{E} Z_1^1(k) = a^k,$$

$$\mathbb{E} [Z_1^1(k)]^2 = \frac{a^{k-1} (a^{k-1} - 1)}{a - 1} b^2 + a^{2k},$$

$$\mathbb{E} [Z_1^1(k)]^3 = \frac{a^{k-1} (a^{2k} - 1)}{a^2 - 1} \gamma +$$

$$+ \frac{3a^{k-1} (a^k - 1) (a^{k-1} - 1)}{(a - 1) (a^2 - 1)} (b^2 + a^2 - a)^2 +$$

$$+ \frac{3a^{k-1} (a^k - 1)}{a - 1} (b^2 + a^2 - a) + a^k. \quad (4)$$

Now using the expansion $\ln x = (x - 1) - \frac{1}{2}(x - 1)^2 + O((x - 1)^3)$, $x \rightarrow 1$ and (4), taking into account that $O((f_{n-k}(t) - 1)^3) = O(|t|^3)$, $t \rightarrow 0$, we get

$$\begin{aligned} \ln \left[e^{-it \frac{A(a,n)}{B(a,n)}} \prod_{k=1}^n f_{n-k}^{\varepsilon_k} \left(\frac{t}{B(a,n)} \right) \right] = \\ = iI_1(n) + I_2(n) + O(I_3(n)) \end{aligned} \quad (5)$$

where

$$I_1(n) = \frac{it}{B(a,n)} \sum_{k=1}^n a^{n-k} (\varepsilon_k - \alpha(k)),$$

$$I_2(n) = \frac{t^2}{2B^2(a,n)} \sum_{k=1}^n \varepsilon_k \frac{a^{n-k} (a^{n-k} - 1)}{a(a - 1)} b^2,$$

$$I_3(n) = \frac{t^3}{B^3(a, n)} \sum_{k=1}^n \varepsilon_k \left\{ \frac{a^{k-1}(a^{2k}-1)}{a^2-1} \gamma + \right. \\ \left. + \frac{3a^{k-1}(a^k-1)(a^{k-1}-1)}{(a-1)(a^2-1)} (b^2 + a^2 - a)^2 + \right. \\ \left. + \frac{3a^{k-1}(a^k-1)}{a-1} (b^2 + a^2 - a) + a^k \right\} + \\ + \frac{|t|^3}{B^3(a, n)} \sum_{k=1}^n a^{3(n-k)} \alpha(k).$$

Note that the equality in (5) is understood in almost surely sense. From [38], Lemma 3.2 we know that if $a < 1$, $b^2 > 0$ and $\alpha(n) \rightarrow \infty$, $n \rightarrow \infty$ then, as $n \rightarrow \infty$

$$(i) \quad A(a, n) \sim \frac{\alpha(n)}{1-a}, \quad (ii) \quad \sigma^2(a, n) \sim \frac{\beta(n)}{1-a^2}, \\ (iii) \quad \Delta^2(a, n) \sim \frac{b^2 \alpha(n)}{(1-a^2)(1-a)}. \quad (6)$$

Thus from assumptions of Theorem 3.1 and (6) it entails that dominated term of $B^2(a, n)$ is $\Delta^2(a, n)$, i.e.,

$$B^2(a, n) \sim \Delta^2(a, n) \quad \text{as } n \rightarrow \infty. \quad (7)$$

We treat each term of right hand side of (5) separately. First we consider $I_1(n)$. By Cauchy-Bunyakovsky inequality, the inequality $2xy \leq x^2 + y^2$ and (6) and also using the properties of regularly varying functions at infinity, we obtain

$$\mathbb{E}[I_1(n)]^2 = \frac{t^2}{B^2(a, n)} \sum_{k=1}^n a^{2(n-k)} \beta(k) + \\ + \frac{2t^2}{B^2(a, n)} \sum_{k=1}^{n-1} \sum_{j=k+1}^{(k+m-1) \wedge n} a^{2n-j-k} \text{cov}(\varepsilon_k, \varepsilon_j) \leq \\ \leq \frac{t^2 \beta(n)}{B^2(a, n)} \sum_{k=1}^n a^{2(n-k)} + \\ + \frac{2t^2}{B^2(a, n)} \sum_{k=1}^{n-1} \sum_{j=k+1}^{(k+m-1) \wedge n} a^{2n-j-k} \sqrt{\beta(j)} \sqrt{\beta(k)} \leq \\ \leq \frac{t^2 \beta(n)}{B^2(a, n)} \frac{a^{2n}-1}{a^2-1} +$$

$$+ \frac{t^2}{B^2(a, n)} \sum_{k=1}^{n-1} \sum_{j=k+1}^{(k+m-1) \wedge n} a^{2n-j-k} (\beta(j) + \beta(k)) \leq \\ \leq \frac{t^2 \beta(n)}{B^2(a, n)} \frac{a^{2n}-1}{a^2-1} + \\ + \frac{t^2}{B^2(a, n)} \sum_{k=1}^{n-1} \sum_{j=k+1}^{(k+m-1) \wedge n} a^{2n-j-k} \beta(k) + \\ + \frac{t^2}{B^2(a, n)} \sum_{k=1}^{n-1} \sum_{j=k+1}^{(k+m-1) \wedge n} a^{2n-j-k} \beta(j) \leq \\ \leq \frac{t^2 \beta(n)}{B^2(a, n)} \frac{a^{2n}-1}{a^2-1} + \\ + \frac{t^2}{B^2(a, n)} \beta(n) (m-1) \sum_{k=1}^n a^{2(n-k)} + \\ + \frac{t^2}{B^2(a, n)} \beta(n) (m-1) \sum_{k=1}^n a^{2(n-k)} \sim \\ \sim \frac{Ct^2(2m-1)}{\Delta^2(a, n)} \beta(n) \rightarrow 0, \quad n \rightarrow \infty.$$

Hence, by Chebyshev's inequality, we have

$$I_1(n) \xrightarrow{P} 0, \quad n \rightarrow \infty. \quad (8)$$

Now we consider $I_2(n)$. From (6)-(7), it follows

$$\mathbb{E}I_2(n) = -\frac{t^2}{2B^2(a, n)} \Delta^2(a, n) \rightarrow -\frac{t^2}{2}, \quad n \rightarrow \infty. \quad (9)$$

By the Cauchy-Bunyakovsky inequality and taking into account the properties of regularly varying functions, we find

$$\text{Var}I_2(n) \leq \\ \leq \frac{t^4 \beta(n)}{4B^4(a, n)} \left[\sum_{k=1}^n \frac{a^{2(n-k)} (a^{n-k}-1)^2}{a^2(a-1)^2} b^4 + \right. \\ \left. + \sum_{k=1}^{n-1} \sum_{j=k+1}^{(k+m-1) \wedge n} \frac{a^{2n-2j} (a^{n-j}-1)^2}{a^2(a-1)^2} \sqrt{\beta(j)} \sqrt{\beta(k)} \right] \\ \leq \frac{t^4}{4B^4(a, n)} [C\beta(n) (a^{4n}-1) +$$

$$\begin{aligned}
 & + \sum_{k=1}^{n-1} \sum_{j=k+1}^{(k+m-1) \wedge n} \frac{a^{2n-2j} (a^{n-j} - 1)^2}{a^2 (a-1)^2} (\beta(j) + \beta(k)) \Bigg] \\
 & \leq \frac{(2m-1) \beta(n) b^4 t^4}{4a^2 (a-1)^2 B^4(a, n)} \frac{a^{4n} - 1}{a^4 - 1} \\
 & \leq \frac{C(2m-1) t^4}{B^4(a, n)} \beta(n) \rightarrow 0, \quad n \rightarrow \infty.
 \end{aligned}$$

Hence, from the last relation and by (9) and virtue of the Chebyshev inequality, we get

$$I_2(n) \xrightarrow{P} -\frac{t^2}{2}, \quad n \rightarrow \infty. \quad (10)$$

Further, taking into account the conditions of the Theorem 3.1, we obtain as $n \rightarrow \infty$

$$\begin{aligned}
 & \mathbb{E} |I_3(n)| \leq \\
 & \leq \frac{|t|^3}{B^3(a, n)} \left[\gamma \sum_{k=1}^n a^{n-k} (a^{2(n-k)} - 1) \alpha(k) + \right. \\
 & \left. + \sum_{k=1}^n a^{n-k} (a^{n-k} - 1) (a^{n-k-1} - 1) \alpha(k) + \right] \\
 & + \frac{(b^2 + a^2 - a) |t|^3}{2a(a-1) B^3(a, n)} \sum_{k=1}^n a^{n-k} (a^{n-k} - 1) \alpha(k) + \\
 & + \frac{|t|^3}{6B^3(a, n)} \sum_{k=1}^n a^{n-k} \alpha(k) + \\
 & + \frac{|t|^3}{B^3(a, n)} \sum_{k=1}^n a^{3(n-k)} \alpha(k) \sim \\
 & \sim C \frac{|t|^3}{B^3(a, n)} \alpha(n) + C \frac{|t|^3}{B^3(a, n)} \alpha(n) + \\
 & + \frac{|t|^3}{B^3(a, n)} + \frac{C|t|^3}{6B^3(a, n)} \alpha(n) + \frac{C|t|^3}{B^3(a, n)} \alpha(n) \rightarrow 0.
 \end{aligned}$$

Consequently, by Markov inequality

$$I_3(n) \xrightarrow{P} 0, \quad n \rightarrow \infty. \quad (11)$$

Thus, combining relations (8)-(11), we obtain

$$\ln \left[e^{-it \frac{A(a, n)}{B(a, n)}} \prod_{k=1}^n f_{n-k}^{\varepsilon_k} \left(\frac{t}{B(a, n)} \right) \right] \xrightarrow{P} -\frac{t^2}{2}$$

as $n \rightarrow \infty$. Hence, according to the Lebesgue's majorized convergence theorem (see [44], Theorem 3, Chapter 2, p.234 and Remark, p.330), we may deduce

$$e^{-it \frac{A(a, n)}{B(a, n)}} \Psi_n \left(\frac{t}{B(a, n)} \right) \rightarrow e^{-\frac{t^2}{2}}, \quad n \rightarrow \infty,$$

which finishes the proof of Theorem 3.1. $\square$

The next result provides conditions for validity of CLT for supercritical branching processes $W(n)$. Note that in this case the immigration mean and variance is not assumed to be a regularly varying at infinity.

Theorem 3.2. Assume $a > 1, \gamma < \infty$ and let $\{\varepsilon_n, n \geq 1\}$ be a sequence of m -dependent random variables. If $\chi(a, n) := \sum_{k=1}^n a^{-2k} \alpha(k) \rightarrow \infty$ and $\beta(n) = o(\chi(a, n))$ as $n \rightarrow \infty$ then as $n \rightarrow \infty$

$$\mathbb{P} \left(\frac{W(n) - A(a, n)}{B(a, n)} < x \right) \rightarrow \Phi(x), \quad x \in \mathbb{R}.$$

Remark 1. Note that the condition $\chi(a, n) \rightarrow \infty$ holds if $\liminf \frac{\alpha(n+1)}{\alpha(n)} \geq a^2$. This means that immigration mean tends to infinity fast enough. Clearly, if $\alpha(n) \in R_\alpha$ then $\chi(a, n)$ does not converge to infinity.

Доведення. The proof of Theorem 3.2 follows the lines of Theorem 3.1. Again we use the relation (5), and we have to evaluate each term involved in the right-hand side of (5). Before we start, it should be pointed out the asymptotic behavior of $B^2(a, n)$ as $n \rightarrow \infty$. Observe that under conditions of Theorem 2, we have as $n \rightarrow \infty$,

$$B^2(a, n) \sim \Delta^2(a, n) \sim C a^{2n} \chi(a, n). \quad (12)$$

It is obvious $\mathbb{E}[I_1(n)] = 0$. By the same argument as in proof of relation (8), we have

$$\begin{aligned}
 \mathbb{E}[I_1(n)]^2 & \leq \frac{t^2}{B^2(a, n)} \sum_{k=1}^n a^{2(n-k)} \beta(k) + \\
 & + \frac{2t^2}{B^2(a, n)} \sum_{k=1}^{n-1} \sum_{j=k+1}^{(k+m-1) \wedge n} a^{2n-j-k} \sqrt{\beta(k)} \sqrt{\beta(j)} \leq
 \end{aligned}$$

$$\begin{aligned} &\leq C \frac{a^{2n} t^2}{B^2(a, n)} \max_{1 \leq k \leq n} \beta(k) + \\ &+ \frac{a^{2n} (m-1) t^2}{B^2(a, n)} \max_{1 \leq k \leq n} \beta(k) \sum_{k=1}^n a^{-2k} + \\ &+ \frac{a^{2n} (m-1) t^2}{B^2(a, n)} \sum_{k=1}^n a^{-2k} \beta(k) \sim \\ &\sim C m \frac{t^2}{\chi(a, n)} \max_{1 \leq k \leq n} \beta(k) \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

Hence, we may claim that

$$I_1(n) \xrightarrow{P} 0, \quad n \rightarrow \infty. \quad (13)$$

Let us consider $I_2(n)$. From (12) it follows that

$$\mathbb{E} I_2(n) = -\frac{t^2}{2B^2(a, n)} \Delta^2(a, n) \rightarrow -\frac{t^2}{2} \text{ as } n \rightarrow \infty.$$

Now we estimate the variance of $I_2(n)$. Observe that

$$\begin{aligned} &\text{Var} I_2(n) \leq \\ &\leq \frac{(2m-1)t^4}{4B^4(a, n)} \sum_{k=1}^n \beta(k) \frac{a^{2(n-k)}(a^{n-k}-1)^2}{a^2(a-1)^2} b^4 \sim \end{aligned}$$

$$\sim \frac{C(2m-1)b^4 t^4}{4(a^{2n}\chi(a, n))^2} a^{4n} \max_{1 \leq k \leq n} \beta(k) \rightarrow 0, \quad n \rightarrow \infty.$$

Thus, by Chebyshev's inequality

$$I_2(n) \xrightarrow{P} -\frac{t^2}{2}, \quad n \rightarrow \infty. \quad (14)$$

Concerning $I_3(n)$, we proceed with the same arguments which were made in (11) and we get

$$\mathbb{E} |I_3(n)| \leq C \frac{\theta |t|^3 \alpha(n)}{\chi(a, n) \sqrt{\chi(a, n)}} +$$

$$+ \frac{C |t|^3 \alpha(n)}{\chi(a, n) \sqrt{\chi(a, n)}} + \frac{C |t|^3 \alpha(n)}{\chi(a, n) \sqrt{a^{2n} \chi(a, n)}} +$$

$$+ \frac{C |t|^3}{a^n \sqrt{a^{2n} \chi(a, n)}} + \frac{|t|^3 \alpha(n)}{\chi(a, n) \sqrt{\chi(a, n)}} \rightarrow 0 \quad (15)$$

as $n \rightarrow \infty$. Collecting relations (13)-(15), we obtain the statement of Theorem 3.2. Theorem 3.2 is completely proved. $\square$

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